A Hybridizable Discontinuous Galerkin formulation for Fluid-Structure Interaction

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Distribution Statement A
Approved for public release
Distribution unlimited

Funding: Naval Sea Systems Command, Advanced Submarine Systems Development Office (SEA 073R)

Fifth deal.II Users and Developers Workshop
Texas A&M University, College Station, TX, USA; August 3-7, 2015
Fluid-Structure Interaction (FSI)

The interaction of a deformable structure with a surrounding or internal fluid

Source
Problem statement

High-order accurate FSI calculations are often too computationally expensive for real-world problems

Solution requirements

High-order accurate computations
Reasonable computational expenses

Proposed solution

Develop the first FSI model that uses the state of the art hybridizable discontinuous Galerkin (HDG) method
HDG – best of both worlds

Continuous Galerkin (CG)
- Computationally efficient for low orders
- Utilizes optimized solvers such as static condensation

Discontinuous Galerkin (DG)
- Scales well for high order
- Preserves local conservation of mass

Hybridizable Discontinuous Galerkin (HDG)
HDG discretized domain

- Local solution
- Global solution trace
- Global boundary

Example: 2D 1x2 quadratic elements

Variables: $u, v, e$ - 8 local DOFs per (□, ▲)
- 2 global DOFs per (●)

CG: 120 global DOFs
DG: 144 global DOFs
HDG: 42 global DOFs
+ 72 local DOFs per cell
Fluid-structure interaction modeling

Domain information and boundary conditions

Fluid model  Solid model  Mesh model

“Couple”

Solve and post process

- Fluid Domain
- Solid Domain
- Mesh Domain
- Fluid-Solid Interface
- Fluid Boundary
- Solid Boundary
Linear elastostatics: HDG discretization

**Strong form**

\[-\text{Div} \left( \underline{C} \left[ e \right] \right) = b\]

\[e - \text{Sym}(\text{Grad} \, u) = 0\]

1. Weight by weighting functions \((\tilde{u}, \tilde{e})\)
2. Integrate over cell \((K)\)
3. Integrate by parts > boundary terms \((\Gamma)\)

**Weak form**

\[\int_{K} \text{Grad} \, \tilde{u} \cdot \underline{C} \left( e \right) \, d\Omega - \int_{\Gamma} \tilde{u} \cdot \underline{C} \left( e \right) n \, d\Gamma = \int_{K} \tilde{u} \cdot b \, d\Gamma\]

\[\int_{K} \tilde{e} : e \, d\Omega - \int_{K} \text{Sym}(\tilde{e}) : \text{Grad} \, u \, d\Omega + \int_{\Gamma} \text{Sym}(\tilde{e}) n \cdot (u - \mu) \, d\Gamma = 0\]

**Flux continuity**

\[\int_{\Gamma} \tilde{\mu} \cdot \underline{C} \left( e \right) n \, d\Gamma = 0\]

**Stabilization**

\[\underline{C} \left( e \right) := \underline{C} \left( e \right) - S (u - \mu) \otimes n\]
**Linear elastostatics: HDG discretization**

**Weak form**
\[
\int_K \text{Grad} \, \tilde{u} : \mathbb{C}(e) \, d\Omega - \int_\Gamma \tilde{u} \cdot \widehat{\mathbb{C}(e)} n \, d\Gamma = \int_K \tilde{u} \cdot b \, d\Gamma \\
\int_K \bar{e} : e \, d\Omega - \int_K \text{Sym}(\bar{e}) : \text{Grad} \, u \, d\Omega + \int_\Gamma \text{Sym}(\bar{e}) n \cdot (u - \mu) \, d\Gamma = 0
\]

\[
\widehat{\mathbb{C}(e)} := \mathbb{C}(e) - \mathbb{S} (u - \mu) \otimes n \quad \int_\Gamma \tilde{\mu} \cdot \widehat{\mathbb{C}(e)} n \, d\Gamma = 0
\]

\[
\begin{pmatrix}
\text{local} \times \text{local} & \text{local} \times \text{global} \\
\text{global} \times \text{local} & \text{global} \times \text{global}
\end{pmatrix}
\begin{Bmatrix}
\text{sol}_{\text{local}} \\
\text{sol}_{\text{global}}
\end{Bmatrix}
= \begin{Bmatrix}
\text{RHS}_{\text{local}} \\
\text{RHS}_{\text{global}}
\end{Bmatrix}
\]

Using the Schur complement, any equation of the form
\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
U \\
\Lambda
\end{pmatrix}
= \begin{pmatrix}
F \\
G
\end{pmatrix},
\]

can be rearranged into the following system of equations:
\[
(D - CA^{-1}B)\Lambda = G - CA^{-1}F,
\]
\[
AU = F - B\Lambda.
\]
Linear elastostatics: verification

**h-refinement (Q3)**

![Graph showing h-refinement (Q3)](image)

**p-refinement**

![Graph showing p-refinement](image)

All errors converge at the expected rate of $h^{p+1}$
Manufactured solution function of sines and cosines
Pure Dirichlet boundary conditions
Preliminary results: Sandia geometry

- Upper (fixed) end of test specimen
- (no hole ahead of the notch)
- (hole ahead of the notch)
- Lower (actuated) end of test specimen

Source
- The 2014 Sandia Fracture Challenge (SFC2)
### Preliminary results: Sandia geometry

#### Undeformed mesh
- 7777 cells

#### Deformed solution
- Showing stress

<table>
<thead>
<tr>
<th>Element Order</th>
<th>Local # of DOFs</th>
<th>Increase from Q1</th>
<th>Global # of DOFs</th>
<th>Increase from Q1</th>
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<td>-</td>
<td>63720</td>
<td>-</td>
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<tr>
<td>Q7</td>
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<td>1500%</td>
<td>254880</td>
<td>300%</td>
</tr>
</tbody>
</table>

Size of global system increases minimally, even for high orders!
Nonlinear elastodynamics: governing equations

Restricted to hyperelastic materials
(Saint Venant-Kirchhoff)

Variables: \( u, v, E \) - local
\( \mu \) - global

Weak form

\[
\int_{K} \tilde{u} \cdot \rho \frac{\partial v}{\partial t} \, dK + \int_{K} \text{Grad} \tilde{u} : F \mathcal{C} (E) \, dK - \int_{\Gamma} \tilde{u} \cdot \mathbf{h} \, d\Gamma = \int_{K} \bar{u} \cdot \mathbf{b} \, dK
\]

\[
\int_{K} \bar{E} : E \, dK - \int_{K} \text{Sym} (\bar{E}) : \text{Grad} \mathbf{u} \, dK - \int_{K} \frac{1}{2} \bar{E} : (\text{Grad} \mathbf{u})^\top \text{Grad} \mathbf{u} \, dK + \int_{\Gamma} \text{Sym} (\bar{E}) \mathbf{n} \cdot (\mathbf{u} - \mu) \, d\Gamma = 0
\]

\[
\int_{K} \tilde{v} \cdot \frac{\partial \mathbf{u}}{\partial t} \, dK - \int_{K} \tilde{v} \cdot v \, dK = 0
\]

\[
\int_{\Gamma} \bar{\mu} \cdot \mathbf{h} \, d\Gamma = 0
\]

where \( \mathbf{h} := [F \mathcal{C} (E)] \mathbf{n} - S (\mathbf{u} - \mu) \)

\( F = I + \text{Grad} \mathbf{u} \)
Preliminary results: Turek and Hron CSM3

Source
- Proposal for numerical benchmarking of fluid-structure interaction between an elastic object and laminar incompressible flow, Turek 2006
Navier-Stokes: governing equations

Variables: v, p, L - local
v, ψ - global

Weak form
\[ \int_K \tilde{v} \cdot \frac{\partial v}{\partial t} \, dK - \int_K \text{grad} \tilde{v} : v \otimes v - \nu L \, dK + \int_K \tilde{v} \cdot \text{grad} p_k \, dK + \int_\Gamma \tilde{v} \cdot \hat{h}^* \, d\Gamma = \int_K \tilde{v} \cdot f \, dK \]
\[ \int_K \tilde{L} : L \, dK - \int_K \tilde{L} : \text{grad} v \, dK + \int_\Gamma \tilde{L} n \cdot (v - \nu) \, d\Gamma = 0 \]
\[ -\int_K \text{grad} \tilde{p}_k \cdot v \, dK + \int_\Gamma \tilde{p}_k v \cdot n \, d\Gamma = 0 \]
Flux stabilization
\[ \hat{h} := [v \otimes v - \nu L + pI] n + S (v - \nu) \]
\[ \hat{h}^* := [v \otimes v - \nu L] n + S (v - \nu) \]
Introduce ψ to close the system
\[ \psi := \frac{1}{|K|} \int_K p \, dK \]
Remove the kernel of the gradient
\[ \int_\Omega p \, d\Omega = 0 \]
HDG ALE transform

\[ \mathbf{L} = \text{grad } \mathbf{v} = \text{Grad } \mathbf{v}_f \mathbf{F}_m^{-1} = \mathbf{L}_f, \]

\[ \frac{\partial \mathbf{v}}{\partial t} = \frac{\partial \mathbf{v}_f}{\partial t} - \mathbf{L}_f [\mathbf{v}_m], \]

Auxiliary HDG variables simplify ALE transformation

Integration domain ALE transform

\[ \int_{\Omega_F(t)} f(\mathbf{x}) \, d\mathbf{x} = \int_{\Omega_F(0)} f_f(\mathbf{X}) J_m \, d\mathbf{X}, \]
2D channel flow around a cylinder
Reynolds number Re=100

Source
- [http://www.featflow.de/en/benchmarks/cfdbenchmarking/flow/dfg_benchmark2_re100.html](http://www.featflow.de/en/benchmarks/cfdbenchmarking/flow/dfg_benchmark2_re100.html)
Preliminary results: flow over cylinder

Matching HDG Navier-Stokes with DFG benchmark

ALE HDG Navier-Stokes proof of concept: oscillating cylinder
Fluid-Structure interaction modeling

Domain information and boundary conditions

Fluid model → Solid model → Mesh model

“Couple”

Solve and post process

Monolithic Coupling

\[
\Omega_S \quad \mu_s \quad v_s \quad T_s \quad n_s \quad \Omega_F \quad \mu_m \quad v_f \quad T_f \quad n_f
\]

\[\Gamma_{FS}\]
Proposed HDG FSI formulation

Problem 1 (Fluid sub-problem)
Find \( \{ \mathbf{L}_h^b, \mathbf{v}_h^b, \mathbf{p}_h^b, \psi_h^b \} \) in \( \mathcal{L}_h^b \times V_h^b \times Q_h^b \times \psi_h^b \) such that

\[
\begin{align*}
\left( \bar{v}_i, \rho J_m \frac{\partial \mathbf{v}_h^b}{\partial t} \right)_{K_i} &= + \left( \mathbf{v}_h^b, \mathbf{L}_h^b \mathbf{v}_h^b - \mathbf{m}_h \mathbf{v}_h^b \right)_{K_i} + \mathbf{F}_m \left[ \begin{array}{c} \text{Grad} \mathbf{p}_h^b \end{array} \right]_{K_i}, \\
\left( \bar{L}_h, J_m \mathbf{L}_h^b \right)_{K_i} &= - \left( \bar{L}_h, J_m \text{Grad} \mathbf{v}_h^b \right)_{K_i} + \left( \bar{L}_h \mathbf{F}_m^\dagger [\mathbf{n}] + J_m (\mathbf{v}_h^b - \mathbf{u}_h^b) \right)_{\partial K_i} = 0,
\end{align*}
\]

Such that

\[
\begin{align*}
\left( \bar{v}_i, J_m \mathbf{F}_m^\dagger [\mathbf{n}] \right)_{\partial T_{i\gamma}} &= 0,
\left( \mathbf{p}_h^b, J_m \mathbf{p}_h^b \right)_{\partial T_{i\gamma}} &= 0,
\end{align*}
\]

\( \forall \{ \mathbf{L}_h, \mathbf{v}_h^b, \mathbf{p}_h^b, \psi_h^b \} \in \mathcal{L}_h^b \times V_h^b \times Q_h^b \times \psi_h^b \), where

\[
\begin{align*}
\mathbf{p}_h^b + \mathbf{p}_h^b &= \mathbf{p}_h, \\
\mathbf{T}_h^b &= - \left( - \mu J_m \mathbf{L}_h^b \mathbf{F}_m^\dagger [\mathbf{n}] + \mathbf{S}_f (\mathbf{v}_h^b - \mathbf{u}_h^b) \right), \\
\mathbf{T}_h^n &= \mathbf{F}_m \left[ \begin{array}{c} \text{Grad} \mathbf{p}_h^b + \mathbf{p}_h^b + \psi_h^b \end{array} \right]_{\partial T_{i\gamma}} + \mathbf{S}_f (\mathbf{v}_h^b - \mathbf{u}_h^b), \\
\mathbf{S}_f &= \frac{\mu}{\rho} \mathbf{I}.
\end{align*}
\]

Problem 2 (Solid sub-problem)
Find \( \{ \mathbf{u}_h^b, \mathbf{v}_h^b, \mathbf{E}_h^b, \mathbf{\mu}_h^b, \psi_h^b \} \) in \( U_h^b \times V_h^b \times \mathbf{E}_h^b \times M_h^b \times \psi_h^b \), such that

\[
\begin{align*}
\left( \bar{v}_i, \frac{\partial \mathbf{v}_h^b}{\partial t} \right)_{K_i} &= + \left( \mathbf{v}_h^b, \mathbf{F}_h \mathbf{C}_m (\mathbf{E}_h^b) \right)_{K_i} + \left( \mathbf{v}_h, \mathbf{T} [\mathbf{n}]_{\partial K_i} = (\mathbf{v}_h, \mathbf{b})_{K_i}, \\
\left( \mathbf{E}_h^b, \mathbf{E}_h^b \right)_{K_i} &= - \left( \text{Sym} (\mathbf{E}_h^b), \text{Grad} \mathbf{u}_h^b \right)_{K_i} - \left( \frac{1}{2} \mathbf{E}_h^b, \text{Grad} \mathbf{u}_h^b \right)_{K_i}, \\
\left( \mathbf{u}_h, \mathbf{u}_h^b - \mathbf{\mu}_h^b \right)_{\partial K_i} &= 0,
\end{align*}
\]

\( \forall \{ \mathbf{u}_h, \mathbf{v}_h, \mathbf{E}_h^b, \mathbf{\mu}_h^b, \psi_h^b \} \in U_h^b \times V_h^b \times \mathbf{E}_h^b \times M_h^b \times \psi_h^b \), where

\[
\begin{align*}
\mathbf{F}_h &= \mathbf{I} + \text{Grad} \mathbf{u}_h^b, \\
\mathbf{T} [\mathbf{n}] := - \left( \mathbf{F}_h \mathbf{C}_m (\mathbf{E}_h^b) \right) \mathbf{n}_h + \mathbf{S}_m (\mathbf{v}_h^b - \mathbf{u}_h^b), \\
\mathbf{S}_m &= \frac{\mu}{\rho} \mathbf{I}.
\end{align*}
\]

Problem 3 (Mesh sub-problem)
Find \( \{ \mathbf{u}_h^b, \mathbf{F}_m^h, \mathbf{\mu}_m^h \} \) in \( U_h^b \times \mathbf{F}_m^h \times M_h^b \), such that

\[
\begin{align*}
\left( \text{Grad} \mathbf{u}_m, \mathbf{C}_m (\mathbf{F}_m^h) \right)_{K} - \left( \mathbf{u}_m, \mathbf{T} [\mathbf{n}]_m \right)_{\partial K} &= \left( \text{Grad} \mathbf{u}_m, \mathbf{C}_m (\mathbf{I}) \right)_{K} - \left( \mathbf{u}_m, \mathbf{C}_m (\mathbf{I}) \mathbf{n}_m \right)_{\partial K} + (\mathbf{u}_m, \mathbf{b})_K, \\
\left( \mathbf{F}_m^h, \mathbf{F}_m^h \right)_{K} - \left( \mathbf{F}_m^h, \text{Grad} \mathbf{u}_m^h \right)_{K} + \left( \mathbf{F}_m^h \mathbf{u}_m - \mathbf{u}_m^h \right)_{\partial K} &= \left( \mathbf{F}_m \mathbf{I}, \mathbf{I} \right)_K, \\
\left( \mathbf{\mu}_m, \mathbf{T} [\mathbf{n}]_m \right)_{\partial T_{i\gamma}} &= \left( \mathbf{\mu}_m, \mathbf{C}_m (\mathbf{I}) \mathbf{n}_m \right)_{\partial T_{i\gamma}} + \left( \mathbf{\mu}_m, \mathbf{g}_m \right)_{\partial T_{i\gamma}} ,
\end{align*}
\]

\( \forall \{ \mathbf{u}_m, \mathbf{F}_m^h, \mathbf{\mu}_m^h \} \in U_m \times \mathbf{F}_m^h \times M_m^b \), where

\[
\begin{align*}
\mathbf{T} [\mathbf{n}]_m := \mathbf{C}_m (\mathbf{F}_m^h) \mathbf{n}_m - \mathbf{S}_m (\mathbf{u}_m^h - \mathbf{u}_m^h), \\
\mathbf{S}_m &= \frac{\mu}{\rho} \mathbf{I}.
\end{align*}
\]

Monolithic Coupling
Actually implementing this in deal

- Multiple Triangulations (for each material)
  - and sets of dofhandlers, fe-sytems, etc

- Using blockmatrix system
  - Each diagonal block gets a different physics model

- Fluid-solid interface (off-diagonal blocks)
  - We have two sets of global (trace) degrees of freedom unlike everywhere else in the domain
    - Face/point tracking between abutting trias
    - Use weak form of coupling conditions with single point quadrature from the adjacent tria
  - Need global-global coupling!
    - Don’t want to couple the local and global solutions
Turek-Hron FSI benchmark

<table>
<thead>
<tr>
<th>parameter</th>
<th>symbol</th>
<th>value [m]</th>
<th>property</th>
<th>value</th>
</tr>
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<tbody>
<tr>
<td>Channel length</td>
<td>$L$</td>
<td>2.5</td>
<td>$\rho_s \frac{[kg]}{[m^3]}$</td>
<td>10000</td>
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<tr>
<td>Channel height</td>
<td>$H$</td>
<td>0.41</td>
<td>$\nu_s$</td>
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<tr>
<td>Cylinder center</td>
<td>$C$</td>
<td>(0.2, 0.2)</td>
<td>$E_s [\nu 6 \frac{[kg]}{[m^3s^2]}]$</td>
<td>5.6</td>
</tr>
<tr>
<td>Cylinder radius</td>
<td>$r$</td>
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<td>$\rho_f [\frac{[kg]}{[m^3]}]$</td>
<td>1000</td>
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<tr>
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<td>$\mu_f [\frac{[kg]}{[m^3s]}]$</td>
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</tr>
<tr>
<td>Flag height</td>
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<td>$\bar{U} [\frac{m}{s}]$</td>
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<td>Reference point</td>
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<td>$\nu_f</td>
<td>_{in} [\frac{[m]}{s}]$</td>
</tr>
<tr>
<td>Reference point</td>
<td>$B$</td>
<td>(0.15, 0.2)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Mesh: 2900 Cells
**Turek-Hron FSI benchmark tip displacement**

Plotted against CG and HDG results, both using backward Euler time-stepping on same mesh

- Q1 HDG results match oscillation period for Q1 CG results
- Q1 HDG results are closer in amplitude to Q2 CG results than Q1
• **Goal:** Develop new computational tool, a HDG FSI model
  – HDG allows for arbitrarily high order polynomials and their inherently parallel nature leads to significant increase in computational efficiency

• **Progress:** Models implemented, tested, and coupled
  – Implemented and verified HDG elasticity and Navier-Stokes models
  – Considered preliminary test cases
  – Implemented HDG FSI formulation

• **HDG FSI:** Benchmarked against Turek and Hron (backward Euler in time)
  – Q1 HDG results match oscillation period for Q1 CG results
  – Q1 HDG results are closer in amplitude to Q2 CG results than Q1
Future research tasks / questions

• Implement higher order time-stepping, such as Diagonally implicit Runge-Kutta

• Perform verification with the method of manufactured solutions for the fully-coupled FSI problem.

• Implement superconvergent postprocess capabilities for HDG FSI as done for individual components.

• Are the various HDG formulations for the different physics models algebraically equivalent? Are there optimal formulations in terms of accuracy, computational efficiency, and/or stability?

• Can we quantify the effects on convergence and dissipation of varying the stability parameter $S$?

• Plotting error vs. wall time, at what order element does using the HDG method for FSI become more computationally efficient than using other methods, such as our CG FSI model?
Thank you

Questions?