

# Nonlinear biharmonic equation and very weak Laplace problem

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*deal.II* - Workshop





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- 2 Nonlinear biharmonic equation
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1 Introduction

2 Nonlinear biharmonic equation

3 Very weak Laplace problem

- We consider the stream function approach to the solution of the stationary Navier-Stokes equations in two dimensions. This leads to a nonlinear biharmonic equation for the stream function.
- Starting with the Navier-Stokes equations one can build the Poisson equation for the pressure.
- Corner singularities restrict the regularity of the stream function and hence the pressure. A very weak formulation of the Poisson equation is required.

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# Derivation of the streamfunction formulation (1)

The Navier-Stokes equations in two dimensions:

For given data  $f$ ,  $v^\partial$  and  $\nu$  find  $\{v, p\} \in H^1(\Omega)^2 \times L^2(\Omega)$ , such that

$$\begin{aligned}\nu(\nabla v, \nabla \varphi) + ((v \cdot \nabla)v, \varphi) - (p, \nabla \cdot \varphi) &= (f, \varphi) && \text{for all } \varphi \in H_0^1(\Omega)^2 \\ (\nabla \cdot v, q) &= 0 && \text{for all } q \in L^2(\Omega)\end{aligned}$$

- We impose Dirichlet boundary conditions on  $\partial\Omega \setminus \Gamma_{\text{out}}$  for the velocity  $v$ .
- On  $\Gamma_{\text{out}}$  we don't prescribe any boundary conditions. There we find the natural condition

$$\nu \partial_n v - pn = 0,$$

which arranges the pressure  $p$ .

## Derivation of the streamfunction formulation (2)

Helmholtz decomposition yields:

- There exists a scalar stream function  $\Psi$  corresponding to the velocity  $v$ , such that  $\text{curl}(\Psi) = v$  is valid.
- Similarly for each test function  $\varphi$  exists a scalar test function  $\Phi$  such that  $\text{curl}(\Phi) = \varphi$  is valid.

For these new variables the problem is:

Find  $\Psi \in H^2(\Omega)$ , such that for all  $\Phi \in H_0^2(\Omega)$  holds

$$\nu(\nabla \text{curl} \Psi, \nabla \text{curl} \Phi) + ((\text{curl} \Psi \cdot \nabla) \text{curl} \Psi, \text{curl} \Phi) = (f, \text{curl} \Phi). \quad (1)$$

The corresponding boundary conditions for the stream function result from the boundary conditions for the velocity  $v$  using the relation  $\text{curl} \Psi = v$ .

$$\Psi^\partial(s) = \Psi^\partial(a) + \int_a^s v_n^\partial(s) ds, \quad (\partial_n \Psi)^\partial(s) = v_t^\partial(s) \quad \text{on } \partial\Omega.$$



# Bogner-Fox-Schmidt element

We solve the stream function problem with a  $H_0^2(\Omega)$ -conforming element (implemented in *deal.II* by *Bärbel Janssen*).

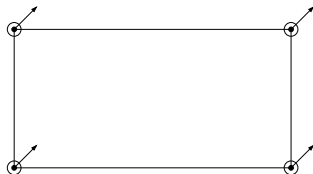


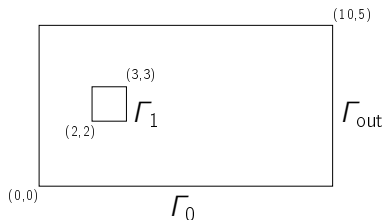
Figure:  $C^1$  element

## Polynomial and ansatz space

$$P(T) = Q_3(T)$$

$$V_h = \{\Phi_h \in L^2(\Omega) : \Phi_h|_T \in P(T), \Phi_h, \nabla\Phi_h \text{ and } \partial_1\partial_2\Phi_h \text{ continuous in corners of } T, \Phi_h = \partial_n\Phi_h = \partial_1\partial_2\Phi_h = 0 \text{ in corners on } \partial\Omega\}.$$

# Flow channel configuration (1)



**Figure:** Configuration with an obstacle,  $\Omega = ((0, 10) \times (0, 5)) \setminus [2, 3]^2$

We impose Dirichlet boundary conditions for  $v$  with a parabolic flow profile. We describe the outer and inner boundary with  $\Gamma_0$  bzw.  $\Gamma_1$ . The conditions for  $\Psi$  are:

$$\begin{aligned}\Psi(x, y) &= -\frac{y^3}{3} + \frac{5y^2}{2}, & \text{on } \Gamma_0 \setminus \Gamma_{\text{out}}, \\ \partial_t \Psi(x, y) &= 0 & \text{on } \Gamma_1 \\ \partial_n \Psi(x, y) &= 0 & \text{on } (\Gamma_0 \setminus \Gamma_{\text{out}}) \cup \Gamma_1\end{aligned}$$

## Flow channel configuration (2)

For implementing the condition on the inner part of the boundary we use the `ConstraintMatrix` class (in *deal.II*). The dofs on the inner boundary are set to be equal.

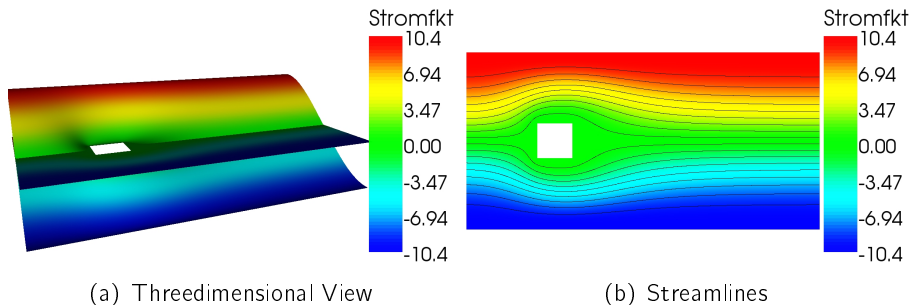
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```
unsigned int help = first_boundary_dof;
for (unsigned int i=first_boundary_dof+1;
     i<dof_handler.n_dofs(); ++i)
{
    if (boundary_dofs[i] == true && i%4==0)
    {
        constraints.add_line (i);
        constraints.add_entry (i, help, 1);
        help=i;
    }
}
```

---

## Flow channel configuration (3)

We linearize with Newton's method. In each Newton step we solve the unsymmetric system with the *biCGstab* method. For preconditioning we use geometric multigrid which is implemented in *deal.II*. By convenience the data is set  $f \equiv 0$  and  $\nu = 1$ .



**Figure:** Stream function on 802816 cells with 3219968 dofs

# Regularity of the stream function

Following *Blum* and *Rannacher* the weak solution of the stream function problem can be written as

$$\Psi(r, \phi) = \sum_{\substack{1 < \operatorname{Re} z_k < 3 \\ q=0, \dots, m_k-1}} a_k r^{z_k} \ln^q r \Psi_k(\phi) + w(r, \phi)$$

with a regular part  $w \in C^3(\overline{\Omega})$  and singular parts

$$a_k r^{z_k} \ln^q r \Psi_k(\phi). \quad (2)$$

The  $z_k$  are certain complex zeros in the interval  $(1, 3)$  with multiplicities  $m_k$ . In equation (2) the term  $r^{\operatorname{Re} z_k}$  dominates the others. One finds  $\operatorname{Re} z_k \approx 1.5445$  for a maximal inner angle of  $270^\circ$  and with the boundary conditions given above. This leads to the regularity  $\Psi \in H^{2.5445}(\Omega)$ .

# Results, error estimates and convergence table

Error analysis gives us the estimates

$$\|\Psi - \Psi_h\|_2 \leq ch^{0.5445} \|\Psi\|_{2.5445},$$

$$\|\Psi - \Psi_h\|_1 \leq ch^{1.089} \|\Psi\|_{2.5445},$$

$$\|\Psi - \Psi_h\|_0 \leq ch^{1.089} \|\Psi\|_{2.5445}.$$

From the simulation we get

# cells	# dofs	$\ \Psi - \Psi_h\ _2$		$\ \Psi - \Psi_h\ _1$		$\ \Psi - \Psi_h\ _0$	
12544	51264	2.6918	0.57	0.0878	1.21	0.0361	1.10
50176	202880	1.8393	0.55	0.0395	1.15	0.0170	1.09
200704	807168	1.2600	0.55	0.0181	1.12	0.0080	1.09
802816	3219968	0.8636	0.54	0.0084	1.11	0.0038	1.09

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# Pressure Poisson equation (1)

We start with the momentum equation strongly formulated

$$\nabla p = f + \nu \Delta v - (v \cdot \nabla)v. \quad (3)$$

We apply the divergence operator on both sides and get

$$-\Delta p = \operatorname{div}((v \cdot \nabla)v) - \operatorname{div} f = 2\{(\partial_1 \partial_2 \psi)^2 - \partial_1^2 \psi \partial_2^2 \psi\} - \operatorname{div} f =: \tilde{f}, \quad (4)$$

where we used  $\operatorname{div} v = 0$ .

The boundary conditions for this problem follow by multiplying (3) with the tangential unity vector  $\tau$  and integrating over a parametrization of  $\partial\Omega$ .

$$\begin{aligned} p(s) &= p(a) + \int_a^s \tau \cdot \{f + \nu \Delta v - (v \cdot \nabla)v\} ds \\ &= p(a) + \int_a^s \tau \cdot \{f + \nu \Delta \operatorname{curl} \psi - (\operatorname{curl} \psi \cdot \nabla) \operatorname{curl} \psi\} ds, \end{aligned}$$



## Pressure Poisson equation (2)

As seen above we analyzed the corner singularities of the stream function formulation. For the pressure we receive the regularity  $p \in H^{0.5445}$ , for which even the weak formulation of (4) doesn't make any sense.

Following *Berggren* we define the ansatz spaces

$$\begin{aligned} V_h &:= \{v_h \in L^2(\Omega) : v_h \in Q_1(T), v_h \text{ continuous in corners of } T\} \\ \gamma V_h &:= \{v|_{\partial\Omega} : v \in V_h\} \\ V_h^0 &:= \{v \in V_h : v|_{\partial\Omega} = 0\} \end{aligned}$$

Consider the following problem:

Find  $p_h \in V_h$ , such that for all  $v_h \in V_h^0$  holds

$$\begin{aligned} \int_T \nabla p_h \cdot \nabla v_h \, dx &= \int_T \tilde{f} v_h \, dx && \text{for all } T \\ p_h &= P_h^\gamma p && \text{on } \partial\Omega. \end{aligned}$$

With  $P_h^\gamma p$  we describe the  $L^2(\partial\Omega)$  projection of  $p$  on  $\gamma V_h$ .

# Flow channel configuration (1)

As we have seen the pressure boundary values have to be projected on the boundary. Because we get the pressure values by integrating, we have two nested integrals, i.e. we have to integrate between two quadrature points. The way of integration is extremely constrained by the character of the domain.

---

```
DoFHandler<2>::active_cell_iterator
    cell = dof_handler.begin_active(),
    endc = dof_handler.end();
DoFHandler<2>::active_cell_iterator
    cell_c1 = dof_handler_c1.begin_active();
for (; cell->vertex(2) != Point<2>(0,5);
     cell=cell->neighbor(3),
     cell_c1=cell_c1->neighbor(3))
{ ... }
```

---

## Flow channel configuration (2)

We already know where the singularities lie. Integrating past a singularity would cause an error. The values for the pressure on outer and inner boundary are coupled through the momentum equation in the domain.

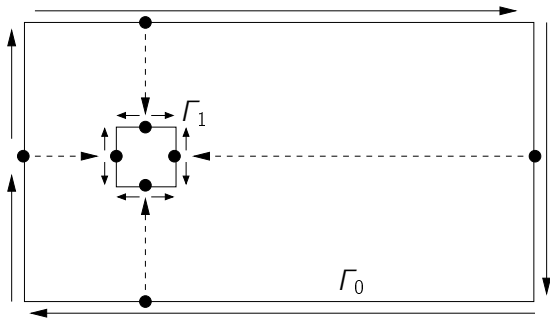


Figure: Integration scheme for the pressure boundary values

## Flow channel configuration (3)

We solve the Poisson equation for the pressure with the CG-method. For preconditioning we use again geometric multigrid.

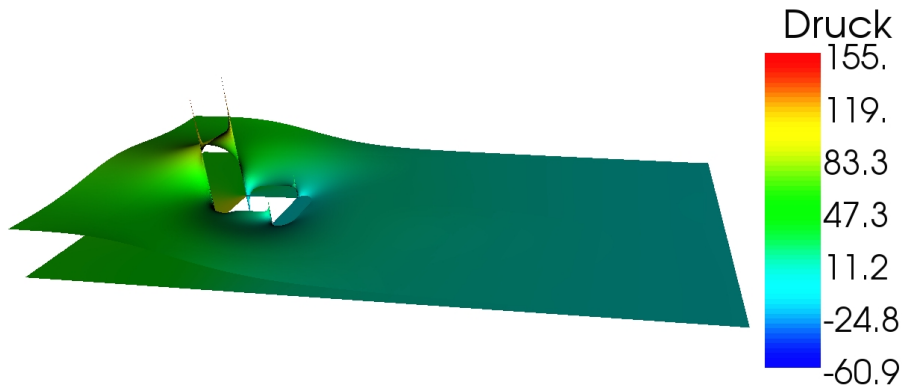


Figure: Pressure on 802 816 cells with 3 215 616 dofs

# Results, error estimate and convergence table

*Berggren* shows in his error analysis

$$\|p - p_h\|_0 \leq ch^{0.167} \|p\|_{0.167}.$$

The simulation yields

# cells	# dofs	$\ p - p_h\ $	
3136	3272	4.28034	0.99
12544	12816	2.65824	0.69
50176	50720	1.83812	0.53
200704	201792	1.29853	0.50
802816	804992	0.91236	0.51

It seems that the error estimate is not optimal.

We try to compare the results with a reference solution of the Navier-Stokes equations solved with the Taylor-Hood element.

	Newton	algebr. MG	geometr. MG	Solver
stream function	×		×	bi-CG-stab
Poisson			×	CG
Navier-Stokes	×	×		GMRES

For solving the Navier-Stokes equations directly, we linearize them with Newton's method. In each Newton step we solve a block system with *GMRES*. The block corresponding to the velocity is inverted with the algebraic multigrid method of *TRILINOS*.

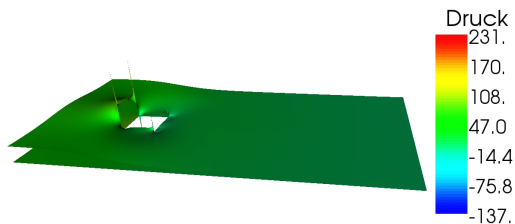
# $H^1$ -error of the pressure

We compare the divergence.

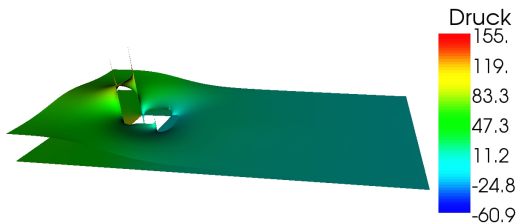
# cells	$\ p - p_h\ _1$ (Poisson)		$\ p - p_h\ _1$ (NS)	
784	61.79549	-	95.2912	-
3136	63.81686	-0.05	112.0709	-0.23
12544	74.97395	-0.23	145.5773	-0.38
50176	95.60175	-0.35	196.1232	-0.43
200704	126.67627	-0.41	267.3613	-0.45
802816	170.79257	-0.43	365.9326	-0.45

The error of the Poisson problem starts with smaller values compared to the error of the direct solution.

# Comparison with the reference solution, graphics



(a) Reference solution with Taylor-Hood



(b) Poisson problem

Figure: Pressure on 802 816 cells



## Comparison with the reference solution, time

# cells	stream function		press via Poisson		NS direct	
	assmbl.	solve	assmbl.	solve	assmbl.	solve
3136	0.8	7.7	0.02	0.0001	1.1	15.7
12544	3.1	55.7	0.06	0.0200	4.5	77.7
50176	12.7	349.4	0.20	0.3960	18.1	403.7
200704	49.3	1444.2	0.67	6.3004	70.2	1769.4
802816	196.1	7835.4	2.53	55.1194	452.7	12754.4

**Table:** Time in sec for assembling the matrix and the right hand side, and for solving the algebraic system

Introducing a stream function will lead to faster calculation than direct solving the Navier-Stokes equations.

- We can solve nonlinear biharmonic equations like the stream function formulation of the Navier-Stokes equations.
- We are able to solve very weak Poisson problems like the pressure Poisson equation.
- Introducing a stream function and recover the pressure through a Poisson equation offers an alternative way to solve the Navier-Stokes equations.

But,

- global refinement is not optimal.
- boundary conditions for the pressure cause errors, for which higher refinement is needed.

Thank you for your time!

