Towards Adaptive FEM for nonlinear ADR Problems

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Outline

Problem statement

$$\frac{\partial u}{\partial t} - div\, a(x, u, \nabla u) + b(x, u, \nabla u) = 0 \quad \text{in } \Omega \times (0, T] \tag{1}$$

$$u = 0 \quad \text{on } \Gamma \times (0, T] \tag{2}$$

$$u = u_0 \quad \text{in } \Omega \tag{3}$$

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$$u = u_0 \quad \text{in } \Omega \tag{3}$$

1. $a(x,u,\nabla u) = D\nabla u, \qquad b(x,u,\nabla u) = c \cdot \nabla u + s(u) - f$
2. $\Omega \subset \mathbb{R}^2$ - a convex polygonal domain with a Lipschitz boundary Γ
3. The final time T is arbitrary
4. $a(x,u,\nabla u)$ and $b(x,u,\nabla u)$ must be continuously differentiable

Weak formulation (theoretical limit)

Solution space

Solution space

$$W^p(a,b;V,W) = \{u \in L^p(a,b;V) : u_t \in L^p(a,b;W)\} \quad (4)$$

with the norm

$$\|u\|_{W^p(a,b;V,W)} = \left\{ \int_a^b \|u(\cdot,t)\|_V^p dt + \int_a^b \|u_t(\cdot,t)\|_W^p dt \right\}^{1/p}, \quad p < \infty \quad (5)$$

Weak formulation (theoretical limit)

Variational Problem

Find $u \in W^r(0,T; W_0^{1,\rho}(\Omega), W^{-1,\pi}(\Omega)), \quad r,\ p,\ \rho,\ \pi \in (1,\infty)$,

such that $\forall v \in L^{p'}(0,T; W_0^{1,\pi'}(\Omega))$:

$$0 = \int_0^T (\partial_t u, v)dt + \int_0^T \{(a(x,u,\nabla u),\nabla v) + (b(x,u,\nabla u),v)\}dt \quad (6)$$

$$u(\cdot,0) = u_0 \quad \text{in } W^{-1,\pi}(\Omega) \quad (7)$$

Weak formulation for our case

- Solution space

$$V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma\} \tag{8}$$

Weak formulation for our case

- Solution space
$$V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma\} \tag{8}$$

- Variational problem:

Find $u \in L^2(0,T;V)$,
such that for almost every $t \in (0,T)$ and $\forall v \in L^2(0,T;V)$:

  - $\partial_t u \in L^2(0,T;H^{-1}(\Omega))$
  - $u(\cdot,0) = u_0 \in H^{-1}(\Omega)$
  - $\int\limits_0^T \{(\partial_t u, v) + (D\nabla u, \nabla v) + (c \cdot \nabla u, v) + (s(u), v)\}dt = \int\limits_0^T (f,v)dt$

Finite element discretization

Theoretical aspects

Notation

1. $N > 0$ – the number of time intervals,
2. $0 = t_0 < t_1 < \dots t_N = T$ – intermediate times
3. $\tau_n = t_n - t_{n-1}$, $1 \leq n \leq N$ – a time step
4. $\mathcal{T}_{h,n}$ – a partition of Ω, assotiated with an intermediate time t_n
5. $X_{h,n}$ – a corresponding finite element space
6. $\widetilde{\mathcal{T}}_{h,n}$ – a shape-regular partition, a refinement of both $\mathcal{T}_{h,n}$ and $\mathcal{T}_{h,n-1}$
7. $\widetilde{\mathcal{E}}_{h,n}$ – the set of all edges of $\widetilde{\mathcal{T}}_{h,n}$

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Assumptions

1. Finite elements are triangles or parallelograms or mixed
2. There are no hanging nodes
3. No anisotropic elements
4. No too abrupt or too strong coarsening

Finite element discretization
θ-scheme

1. Choose a parameter $\theta \in [0.5, 1]$
2. Find $u_h^n \in X_{h,n}$, $0 \le n \le N$, such that

$$u_h^0 = \pi_0 u_0, \tag{9}$$

for $n = 1, \dots, N$, $\forall v_h \in X_{h,n}$:

$$\begin{aligned}(D^{n\theta}\nabla u_h^{n\theta}, \nabla v_h) + (c^{n\theta} \cdot \nabla u_h^{n\theta} + s(u_h^{n\theta}), v_h) + \frac{1}{\tau_n \theta}(u_h^{n\theta}, v_h) = \\ = (f^{n\theta}, v_h) + \frac{1}{\tau_n \theta}(u_h^{n-1}, v_h),\end{aligned} \tag{10}$$

$$u_h^{n\theta} = \theta u_h^n + (1-\theta)u_h^{n-1}. \tag{11}$$

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$$= (f^{n\theta}, v_h) + \frac{1}{\tau_n \theta}(u_h^{n-1}, v_h),$$

$$u_h^{n\theta} = \theta u_h^n + (1-\theta) u_h^{n-1}. \tag{11}$$

$$\begin{aligned} f^{n\theta} &= \theta f(., t_n) + (1-\theta) f(., t_{n-1}) \\ D^{n\theta} &= \theta D(., t_n) + (1-\theta) D(., t_{n-1}) \\ c^{n\theta} &= \theta c(., t_n) + (1-\theta) c(., t_{n-1}) \end{aligned} \tag{12}$$

Spatio-temporal error indicator

$$E_{T,\Omega} = \{\sum_{n=1}^{N} \tau_n [(\eta_h^n)^p + (\widetilde{\eta}_h^n)^p + \|\nabla \widetilde{u}_h^n\|_{0,\pi}^p]\}^{1/p} \tag{13}$$

Spatio-temporal error indicator

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1. $E_\Omega := \eta_h^n$ – spatial error indicator
2. $E_T := ((\widetilde{\eta}_h^n)^p + \|\nabla \widetilde{u}_h^n\|_{0,\pi}^p)^{1/p}$ – temporal error indicator
3. $\widetilde{u}_h^n$ – the solution of the auxiliary discrete Poisson problem
4. $\|u\|_{0,\pi}^p := \int_\Omega |u(x)|^\pi dx$

Spatial error indicator

For every integer n between 1 and N define a spatial error indicator η_h^n by

$$\eta_h^n = \{ \sum_{K \in \widetilde{\mathcal{T}}_{h,n}} h_K^\pi \|R_K\|_{0,\pi;K}^\pi + \sum_{E \in \widetilde{\mathcal{E}}_{h,n}} h_E \|R_E\|_{\pi;E}^\pi \}^{1/\pi} \tag{14}$$

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Element norm

$$\|u\|_{0,\pi;K}^\pi := \int_K |u(x)|^\pi dx \tag{15}$$

Edge norm

$$\|u\|_{\pi;E}^\pi := \int_E |u(x)|^\pi ds(x) \tag{16}$$

Spatial error indicator
Residuals

Element residuals R_K, $K \in \widetilde{\mathcal{T}}_{h,n}$

$$R_K = f^{n\theta} + \frac{1}{\tau_n \theta}\left(u_h^{n-1} - u_h^{n\theta}\right) + div(D\nabla u_h^{n\theta}) - c \cdot \nabla u_h^{n\theta} - s(u_h^{n\theta}) \qquad (17)$$

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Edge residuals R_E, $E \in \widetilde{\mathcal{E}}_{h,n}$

$$R_E = \begin{cases} 0 \quad \text{if} \quad E = \partial K \cap \partial K' \subset \Gamma \\ [n_E \cdot D\nabla u_h^{n\theta}]_E = \frac{1}{2}\left(n_E \cdot D\nabla u_h^{n\theta}\big|_{\partial K} - n_E \cdot D\nabla u_h^{n\theta}\big|_{\partial K'}\right) \text{ if } E \not\subseteq \Gamma, \end{cases} \qquad (18)$$

Auxiliary discrete Poisson problem

$\widetilde{u}_h^n \in S^1(\widetilde{\mathcal{T}}_{h,n})$, $1 \leq n \leq N$ is the unique solution of the discrete Poisson equation

$$\int\limits_{\Omega} \nabla \widetilde{u}_h^n \nabla v_h dx = \langle r_\tau^n, v \rangle \quad \forall v_h \in S^1(\widetilde{\mathcal{T}}_{h,n}) \tag{19}$$

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$$\langle r_\tau^n, v \rangle = \int_\Omega \{\nabla v \cdot D \nabla (u_h^n - u_h^{n-1}) + vc \cdot \nabla (u_h^n - u_h^{n-1})\} dx + \tag{20}$$

$$+ \int_\Omega \{v s_u'(u_h^{n\theta})(u_h^n - u_h^{n-1})\} dx$$

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$$+ \int_\Omega \{v s_u'(u_h^{n\theta})(u_h^n - u_h^{n-1})\} dx$$

1. u_h^n – the solution corresponding to t_n,
2. u_h^{n-1} – the solution on the t_{n-1},
3. $\tau_n = t_n - t_{n-1}$ – current time step.
4. $S^1(\widetilde{\mathcal{T}}_{h,n})$ – the space of continuous piecewise linear finite element functions corresponding to $\widetilde{\mathcal{T}}_{h,n}$

Temporal error estimator

1. Residual-based part

$$\widetilde{\eta}_h^n = \left\{ \sum_{K \in \widetilde{\mathcal{T}}_{h,n}} h_K^\pi \|\widetilde{R}_K + \Delta \widetilde{u}_h^n\|_{0,\pi;K}^\pi + \sum_{E \in \widetilde{\mathcal{E}}_{h,n}} h_E \|[n_E \cdot (\nabla \widetilde{u}_h^n - \widetilde{R}_E)]_E\|_{\pi;E}^\pi \right\}^{1/\pi} \tag{21}$$

Temporal error estimator

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2. Element residual

$$\widetilde{R}_K = -div\left[D\nabla(u_h^n - u_h^{n-1})\right] + c\cdot\nabla(u_h^n - u_h^{n-1}) + s_u'(u_h^{n\theta})(u_h^n - u_h^{n-1}) \tag{22}$$

3. Edge residual

$$\widetilde{R}_E = D\nabla(u_h^n - u_h^{n-1}) \tag{23}$$

4. $\widetilde{u}_h^n$ – the unique solution of the auxiliary discrete Poisson problem.

Input parameters

Prescribe

parameters for spatial adaptation:

1. TOL_{space} – tolerance corresponding to spatial adaptation
2. TOL_{coarse} – tolerance corresponding to spatial coarsening
3. S_{space} – mesh adaptation strategy
4. M_{space} – mesh smoothing strategy

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parameters for time step regulation:

1. $TOL_{time,max}$ – maximal tolerance corresponding to time adaptation
2. $TOL_{time,min}$ – minimal tolerance corresponding to time adaptation
3. $\delta_1 \in (0,1)$
4. $\delta_2 > 1$

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4. $\delta_2 > 1$

u_h^{n-1} – solution at the time t^{n-1} on the mesh M^{n-1} with time step τ_{n-1}

Time and space adaptive algorithm

Step 1.

$M^n := M^{n-1},\ \tau^n := \tau^{n-1},\ t^n := t^{n-1} + \tau^n.$ Solve for u_h^n on M^n

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Step 2.

while $E_T > TOL_{time,max}/(2T)$ do

$\tau^n := \delta_1 \tau^n, t^n := t^{n-1} + \tau^n$

Solve for u_h^n on M^n. Compute E_T on M^n

endwhile

Time and space adaptive algorithm

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Solve for u_h^n on M^n. Compute E_T on M^n

endwhile

Step 3.

while $E_\Omega > TOL_{space}/T$ do

obtain refined mesh M^{n+1} based on $\{E_K\}$ and S_{space}

set $M^n := M^{n+1}$; solve for u_h^n on M^n; execute **Step 2.**

endwhile

Time and space adaptive algorithm

Step 1.

$M^n := M^{n-1}, \; \tau^n := \tau^{n-1}, \; t^n := t^{n-1} + \tau^n.$ Solve for u_h^n on M^n

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while $E_T > TOL_{time,max}/(2T)$ do

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endwhile

Step 4.

Coarsen M^n, producing a modified mesh M^n according to $E_\Omega > TOL_{coarse}/T$

Time and space adaptive algorithm

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Coarsen M^n, producing a modified mesh M^n according to $E_\Omega > TOL_{coarse}/T$

Step 5.

if $E_T < TOL_{time,min}/(2T)$ then $\tau^n := \delta_2 \tau^n$

Why deal.II?

- Pretty good documentation!
- Edge residuals
- Coarsening (which Femlab is not capable of)
- different mesh smoothing strategies

Parameter Values in Experiments

Prescribe

parameters for spatial adaptation:

1. $TOL_{space} = 10$
2. $TOL_{coarse} = 2.5$
3. S_{space} = fixed number $(r = .15, c = 0.)$
4. M_{space} = patch level 1

parameters for time step regulation:

1. $TOL_{time,max} = 10$
2. $TOL_{time,min} = 3.7$
3. $\delta_1 = .5$
4. $\delta_2 = 2$

Linear Test Equation

$$\partial_t u - \nabla^2 u = f(x,t) \qquad \forall x = (x_1, x_2) \in \Omega \times (0, T] \tag{24}$$

$$u(t=0) = u_0 \tag{25}$$

$$u\Big|_{\partial\Omega} = g \tag{26}$$

The inhomogeneity f, spatial domain $\Omega \subset \mathbb{R}^2$, final time T, initial condition u_0 and boundary condition g are specified in the examples.

Example 1 - Gaussian Clock

- Linear test equation
- Desired solution:

$$u(x,t) = e^{-h(x,t)} \tag{27}$$

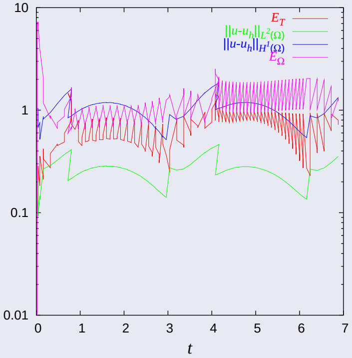
$$h(x,t) = 5\,[x_1 - 2\cos(t)]^2 + [x_2 - 2\sin(t)]^2 \tag{28}$$

- Equation data:

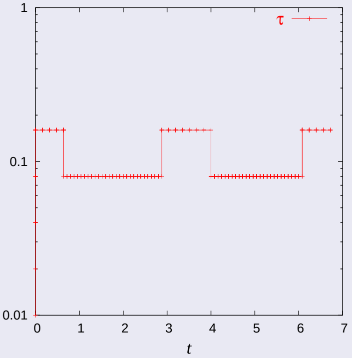
$$\begin{aligned} f &= \partial_t u - \nabla^2 u \\ u_0 &= u(x,0) \\ g &= u(x,t)\Big|_{\partial\Omega\times(0,T]} \\ T &= 7 \end{aligned}$$

Example 1 - Solver Performance

Error history



Time step history



Example 2 - Oscillating Gaussian

- Linear test equation
- Desired solution:

$$u(x,t) = e^{-h(x,t)} \tag{29}$$

$$h(x,t) = \sigma_1(t)\,[x_1 - 2\cos(t)]^2 + \sigma_2(t)\,[x_2 - 2\sin(t)]^2 \tag{30}$$

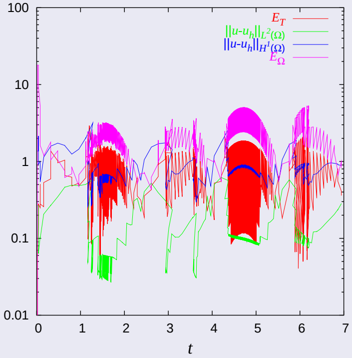
$$\sigma_1(t) = 10\,[1.1 + \cos(4t)] \tag{31}$$

$$\sigma_2(t) = .75\,[1.5 + .5\sin(3t)] \tag{32}$$

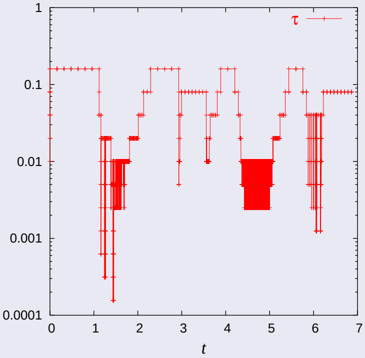
- Equation data:
  Analogously to Example 1.

Example 2 - Solver Performance

Error history



Time step history



Example 3 - Gaussian Clock quadratic elements

- Linear test equation
- Desired solution:

$$u(x,t) = e^{-h(x,t)} \tag{33}$$

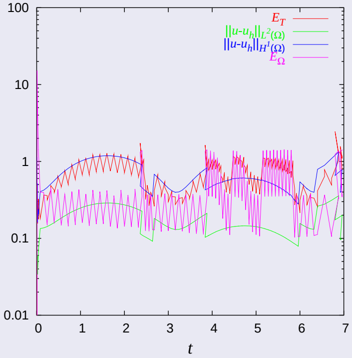
$$h(x,t) = 5\,[x_1 - 2\cos(t)]^2 + [x_2 - 2\sin(t)]^2 \tag{34}$$

- Equation data:

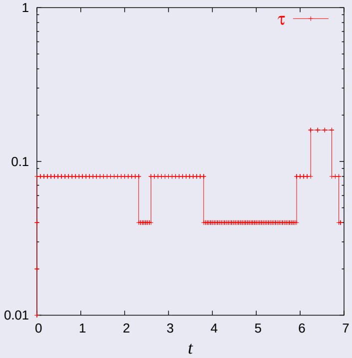
$$\begin{aligned} f &= \partial_t u - \nabla^2 u \\ u_0 &= u(x,0) \\ g &= u(x,t)\Big|_{\partial\Omega\times(0,T]} \\ T &= 7 \end{aligned}$$

Example 3 - Solver Performance

Error history



Time step history



Example 4 - Gaussian Clock cubic elements

- Linear test equation
- Desired solution:

$$u(x,t) = e^{-h(x,t)} \tag{35}$$

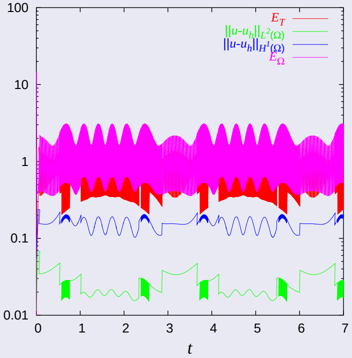
$$h(x,t) = 5\left[x_1 - 2\cos(t)\right]^2 + \left[x_2 - 2\sin(t)\right]^2 \tag{36}$$

- Equation data:

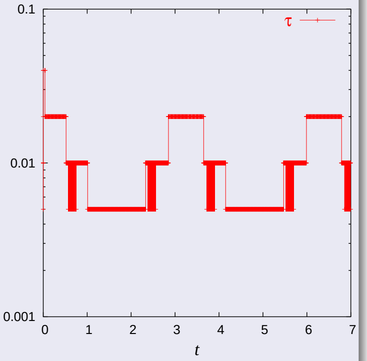
$$\begin{aligned} f &= \partial_t u - \nabla^2 u \\ u_0 &= u(x,0) \\ g &= u(x,t)\Big|_{\partial\Omega\times(0,T]} \\ T &= 7 \end{aligned}$$

Example 4 - Solver Performance

Error history



Time step history



Example 5 - Gaussian Clock cubic elements

- Linear test equation
- Desired solution:

$$u(x,t) = e^{-h(x,t)} \tag{37}$$

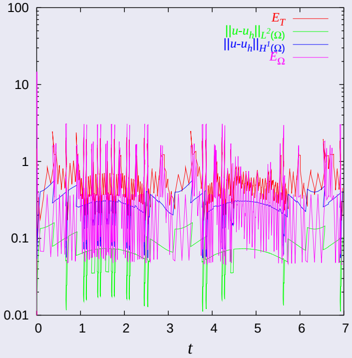
$$h(x,t) = 5\,[x_1 - 2\cos(t)]^2 + [x_2 - 2\sin(t)]^2 \tag{38}$$

- Equation data:

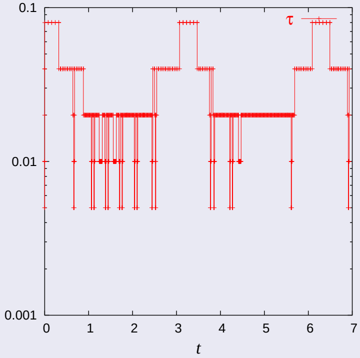
$$\begin{aligned} f &= \partial_t u - \nabla^2 u \\ u_0 &= u(x,0) \\ g &= u(x,t)\Big|_{\partial\Omega\times(0,T]} \\ T &= 7 \end{aligned}$$

Example 5 - Solver Performance

Error history



Time step history



Finally

Thank you for your attention!