



LSFEM for FSI-Problems, 3.1.2006

LSFEM for the Solution of Fluid-Structure Interaction Problems

Oliver Kayser-Herold

Institut für Wissenschaftliches Rechnen
Technische Universität Braunschweig



Least Squares Methods for FSI-Problems

1. FSI-Problem
2. LSFEM Basics
3. Used LSFEM Formulation
4. Implementation with deal.II
5. Example



1. Coupled Problem

Γ_f

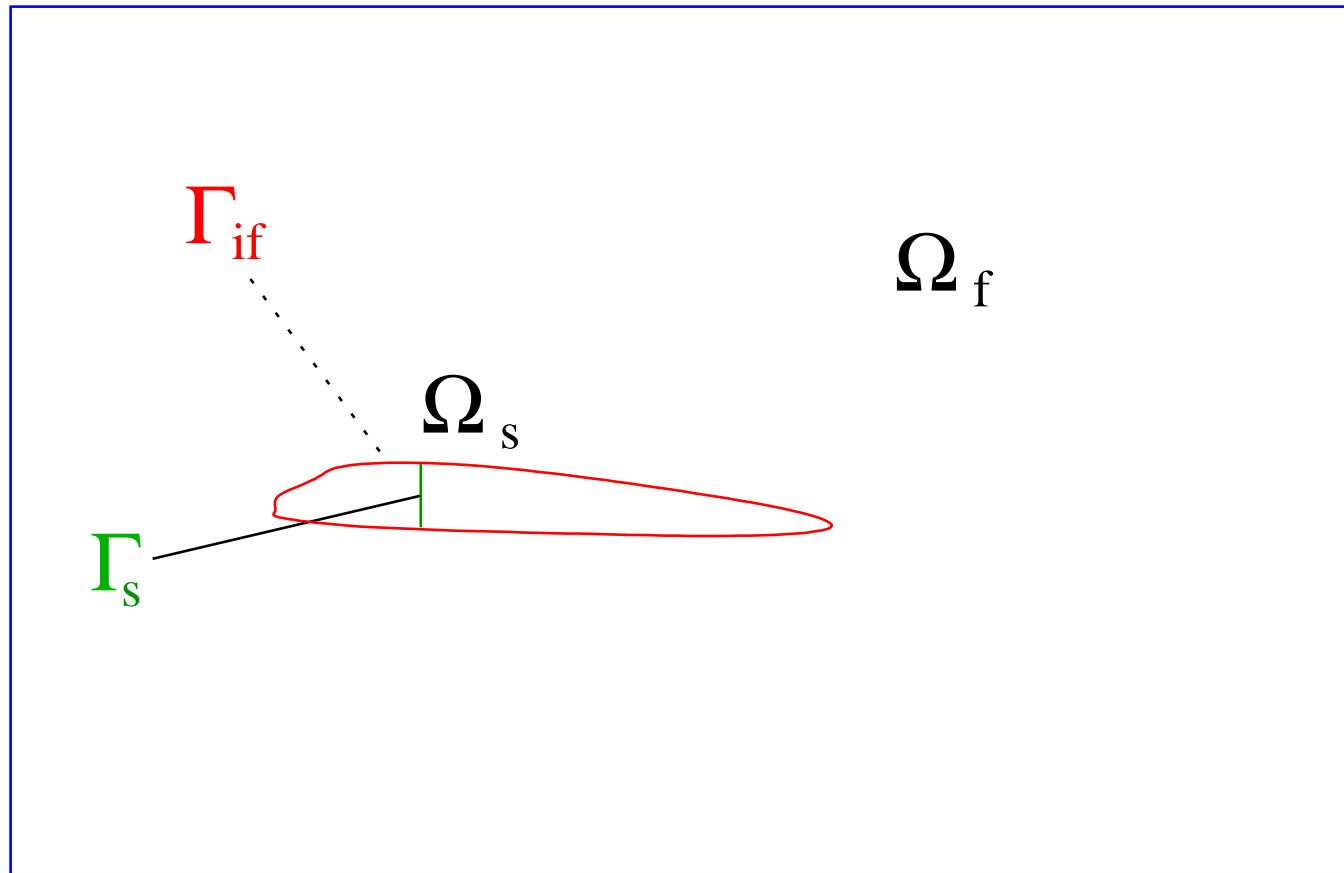


Abbildung 1: Scheme of the model



1. Fluid Structure Problems

Navier-Stokes eqs. for incompressible transient flows

$$\frac{\partial \mathbf{v}_f}{\partial t} + \mathbf{v}_f \cdot \nabla \mathbf{v}_f - \nu_f \Delta \mathbf{v}_f + \frac{1}{\rho_f} \nabla p = \frac{\mathbf{f}}{\rho_f} \quad \text{in } \Omega_f \quad (1)$$

$$\nabla \cdot \mathbf{v}_f = 0 \quad \text{in } \Omega_f \quad (2)$$

Linear elastodynamics

$$\rho_s \frac{\partial \mathbf{v}_s}{\partial t} - \nabla \cdot \boldsymbol{\sigma}_s = \mathbf{f}, \quad \boldsymbol{\sigma}_s = \lambda(\text{tr} \boldsymbol{\varepsilon}) \mathbf{I} + 2\mu \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} = \frac{1}{2}((\nabla \mathbf{u})^T + \nabla \mathbf{u}) \quad \text{in } \Omega_s \quad (3)$$

Coupling conditions

$$\mathbf{v}_s = \mathbf{v}_f, \quad \text{on } \Gamma_{if} \quad (4)$$

$$\boldsymbol{\sigma}_f \cdot \mathbf{n} = \boldsymbol{\sigma}_s \cdot \mathbf{n}, \quad \text{on } \Gamma_{if} \quad (5)$$



2. Weighted Residual Methods

To solve

$$\mathcal{L}u = f \quad \text{in } \Omega \quad (6)$$

$$\mathcal{R}u = g \quad \text{on } \Gamma \quad (7)$$

Method of weighted residual:

$$\text{seek } u \in \mathbf{V} \quad \text{s.t.} \quad \int_{\Omega} (\mathcal{L}u - f)\varphi \, d\Omega = 0 \quad \forall \varphi \in \mathbf{W} \quad (8)$$

The choice of φ determines the numerical properties of the method:

- Galerkin : $\varphi \in \mathbf{V}$
- Finite Volume : $\varphi = \chi_{\Omega_i}$
- **Least Squares** : $\varphi = \mathcal{L}v$ für $\varphi \in \mathbf{W}$



2. Problems of the Galerkin FEM

- Difficulties in the context of saddle point problems (Stokes Eqs.)
- First order differential operators require some stabilisation
- Resulting systems of equations often nonsymmetric

Intensive research resolved/circumvented most of these problems!



2. Basics of the LSFEM

The least squares FEM minimises the following functional:

$$\mathcal{J}(u) = \frac{1}{2}(\|\mathcal{L}u - f\|_a^2 + \|\mathcal{R}u - g\|_b^2) \quad (9)$$

Higher order differential operators cause problems:

$$-\frac{\partial^2 u}{\partial x^2} = 0 \rightarrow \int_{\Omega} \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial x^2} = 0 \quad (10)$$

Best suited for first order systems!



2. Advantages of the LSFEM

Properties of the LSFEM:

- Very robust (no explicit stabilisation of first order terms necessary)
- Resulting systems of equations always symmetric positive definite
- Mathematical theory of the Galerkin FEM applicable in many parts
- In some cases very efficient solution procedures exist (Multigrid with $\mathcal{O}(n)$)



3. LSFEM for the Navier-Stokes Eqs.

By introducing the vorticity ω , the Navier-Stokes eqs. become:

$$\mathcal{L}(\mathbf{v}_f, \omega, p) = \mathbf{v}_f \cdot \nabla \mathbf{v}_f + \nu \nabla \times \omega + \frac{1}{\rho_f} \nabla p - \frac{\mathbf{f}}{\rho_f} = 0 \quad (11)$$

$$\mathcal{C}_1(\mathbf{v}_f) = \nabla \cdot \mathbf{v}_f = 0, \quad (12)$$

$$\mathcal{C}_2(\mathbf{v}_f, \omega) = \omega - \nabla \times \mathbf{v}_f = 0. \quad (13)$$

The corresponding coercive LSFEM functional is ($q \geq 0$):

$$\begin{aligned} \mathcal{J}_f(\mathbf{v}_f, p, \omega) = & \\ & \frac{1}{2} \left(\left\| \mathbf{v}_f \cdot \nabla \mathbf{v}_f + \nu \nabla \times \omega + \frac{1}{\rho_f} \nabla p - \frac{\mathbf{f}}{\rho_f} \right\|_q^2 \right. \\ & \left. + \|\nabla \cdot \mathbf{v}_f\|_{q+1} + \|\omega - \nabla \times \mathbf{v}_f\|_{q+1}^2 \right) \end{aligned} \quad (14)$$



3. $v\sigma$ -Formulation for Elastodynamics, I

Starke et.al. proposed the following first order system for linear elasticity:

$$\mathbb{C}^{-1/2}\sigma - \mathbb{C}^{1/2}\varepsilon(\mathbf{u}) = 0 \quad \text{in } \Omega \quad (15)$$

$$\nabla \cdot \sigma = 0 \quad \text{in } \Omega \quad (16)$$

Reformulating the constitutive law in terms of the strain- and stressrates yields:

$$\mathbb{C}^{-1/2}\dot{\sigma} - \mathbb{C}^{1/2}\varepsilon(\dot{\mathbf{u}}) = 0 \quad \text{in } \Omega. \quad (17)$$

which leads to the following formulation for elastodynamics:

$$\mathbb{C}^{-1/2}\frac{\partial\sigma}{\partial t} - \mathbb{C}^{1/2}\varepsilon(\mathbf{v}) = 0 \quad \text{in } \Omega \quad (18)$$

$$\frac{\partial\mathbf{v}}{\partial t} - \nabla \cdot \sigma = 0 \quad \text{in } \Omega \quad (19)$$



3. $v\sigma$ -Formulation for Elastodynamics, II

Least squares functional:

$$\begin{aligned} \mathcal{J}_s^*(\sigma_{n+1}, \mathbf{v}_{n+1}) = & \frac{1}{2} (\| \mathbb{C}^{-1/2} \sigma_{n+1} - k_1 \mathbb{C}^{1/2} (\nabla \mathbf{v}_{n+1} + (\nabla \mathbf{v}_{n+1})^T) \|_0^2 + \\ & \| \mathbf{v}_{n+1} - k_1 \nabla \cdot \sigma_{n+1} \|_0^2) \end{aligned} \quad (20)$$

The resulting bilinear is only coercive in $H(\text{div}) \times H^1$. Thus finite dimensional subspaces of $H(\text{div})$ have to be used for the approximation of the stress field. Currently RT spaces are used.



3. LSFEM/LSFEM Coupling

Least squares functional for the fully coupled problem:

$$\mathcal{J}_{tot} = \mathcal{J}_f + \mathcal{J}_s + \mathcal{J}_{if} \quad (21)$$

with $\mathcal{J}_{if} = \alpha_C^1 \mathcal{J}_{co} + \alpha_C^2 \mathcal{J}_{tr}$

Compatibility condition:

$$\mathcal{J}_{co} = \frac{1}{2} \|\mathbf{v}_f - \mathbf{v}_s\|_{1/2(\Gamma_{if})}^2 \quad (22)$$

Equivalence of the normal tractions:

$$\mathcal{J}_{tr} = \|(-\mathbf{I}p + 2\nu_f \mathbf{D} - \sigma_s) \mathbf{n}\|_{-1/2(\Gamma_{if})}^2 \quad (23)$$



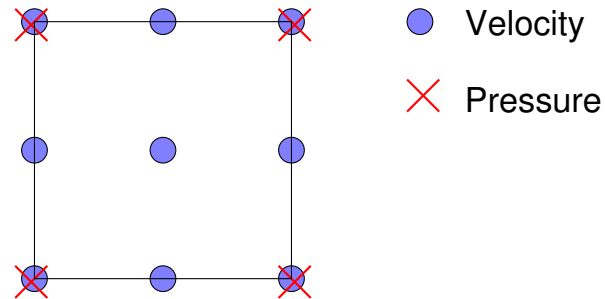
4. Solution Procedure

Evaluation of the functional:

- H^1 -norm is replaced by a weighted L_2 -norm

Elementpairs:

- Q2-Q1
- Q3-Q2
- Q4-Q3



Q2-Q1 Element-Pair

Numerical Solution

- Full Newton iteration
- Direct solver (UMFPack)

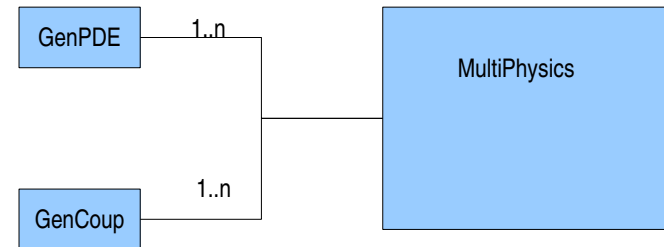
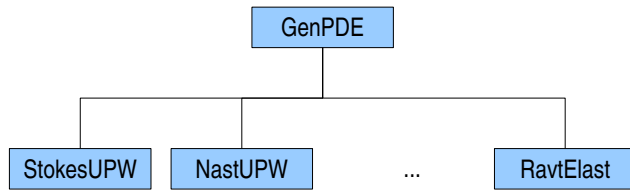


4. Implementation with deal.II framework

- To keep the advantage of strict geometric information, a single `Triangulation` is used for the complete domain.
- Two `DoFHandlers` are used for the fluid and structural part
- Flag `material_id` is used to differentiate fluid and structure cells
- Due to the number of available LSFEM formulations, the solvers were implemented to be “pluggable“ modules.
- Interface implemented as interior boundary



4. Structure of FSI code





4. Modifications of deal.II code

- No interior boundaries are allowed: removed some assertions
- Several methods work on all cells: Adapted to be selective

Standard:

```
template <int dim>
void DoFHandler<dim>::distribute_dofs (const FiniteElement<dim> &ff,
    const unsigned int offset)
```

Modified:

```
template <int dim>
void DoFHandler<dim>::distribute_dofs (const FiniteElement<dim> &ff,
    const unsigned int offset,
    const unsigned char material)
```

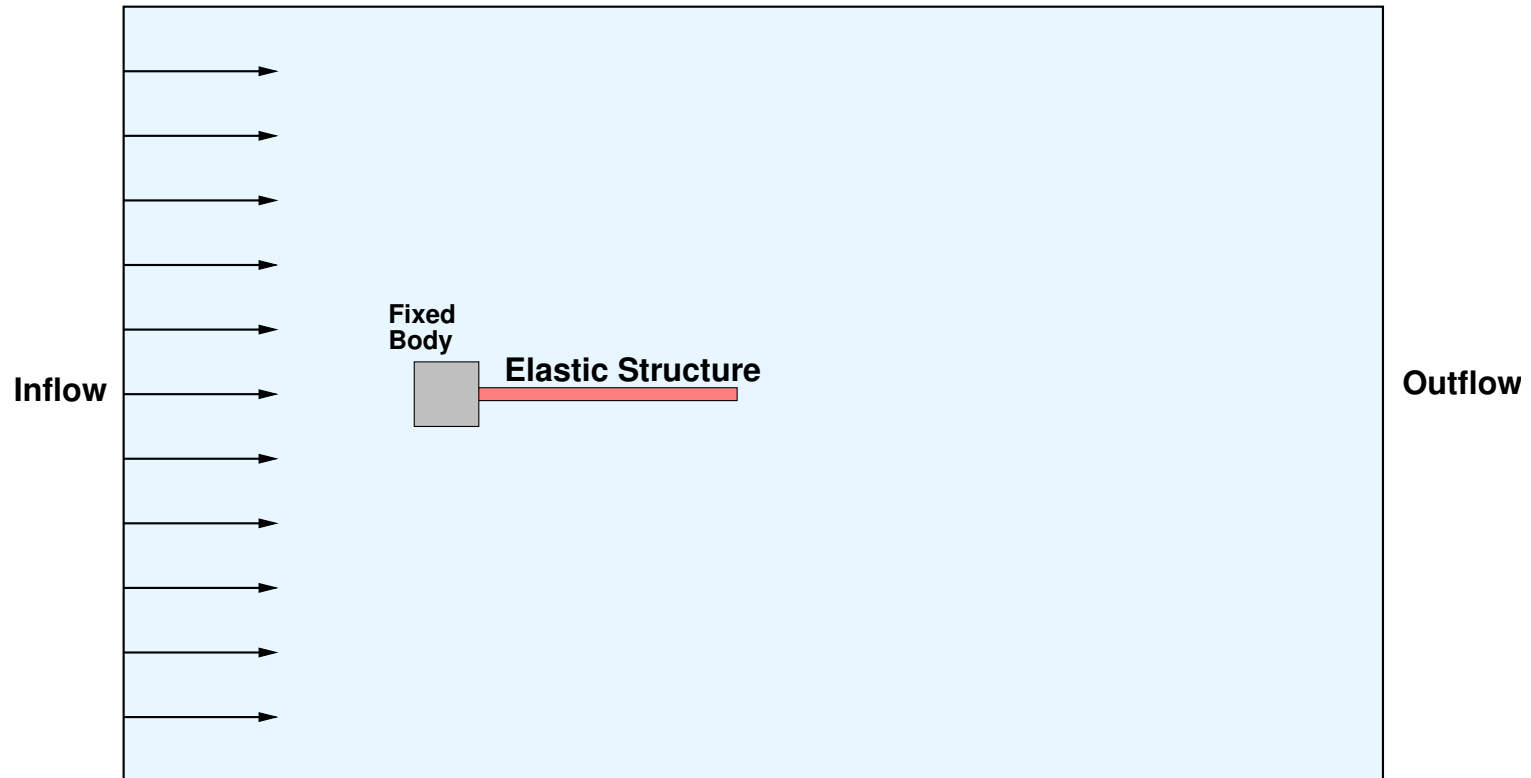



4. Handling of the mesh deformation

- ALE-Formulation for the fluid domain.
- The fluid domain is deformed like an elastic structure.
- MappingQ1Eulerian is used to handle the deformation.
- Mesh deformation is not integrated into NL iterations



5. Example



$$\text{Fluid: } \rho_f = 1.18 \cdot 10^{-3}, \mu_f = 1.82 \cdot 10^{-4}$$

$$\text{Structure: } \rho_s = 0.1, E = 2.5 \cdot 10^6, \mu_s = 0.35$$



5. Results, I

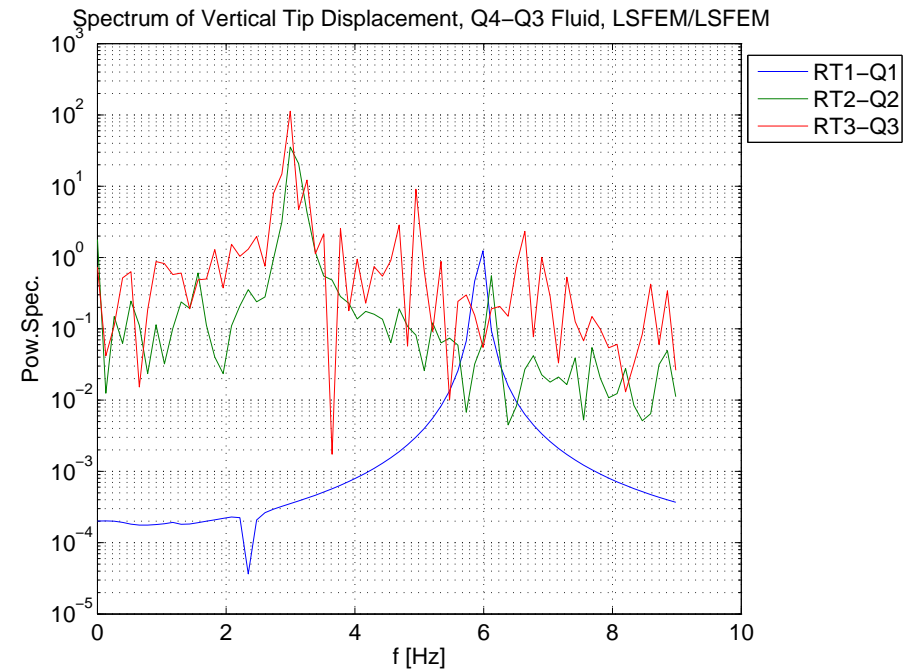
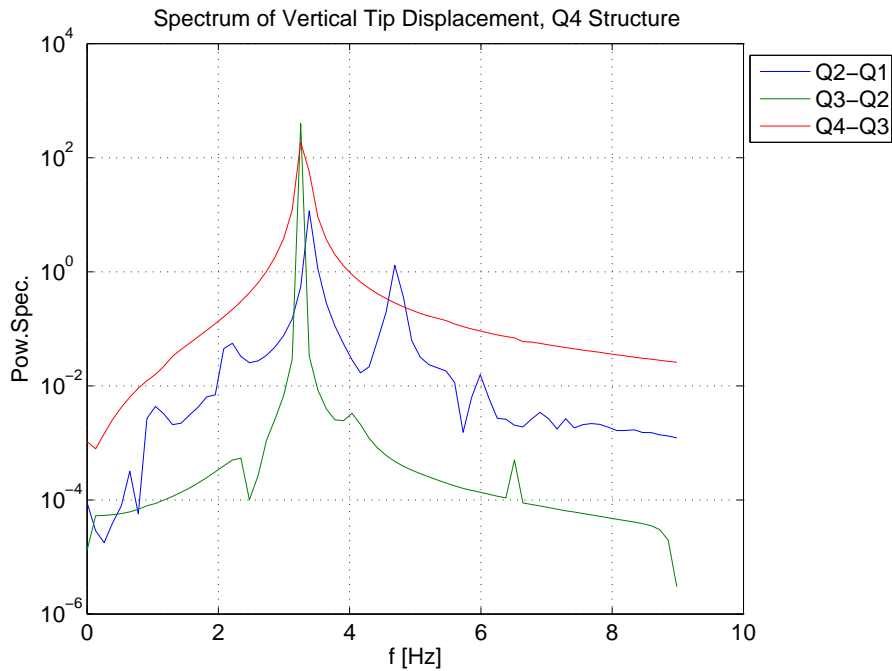


Abbildung 2: Power Spectrum of Tip Disp., Galerkin/LSFEM

Abbildung 3: Power Spectrum of Tip Disp., LSFEM/LSFEM



5. Results, II

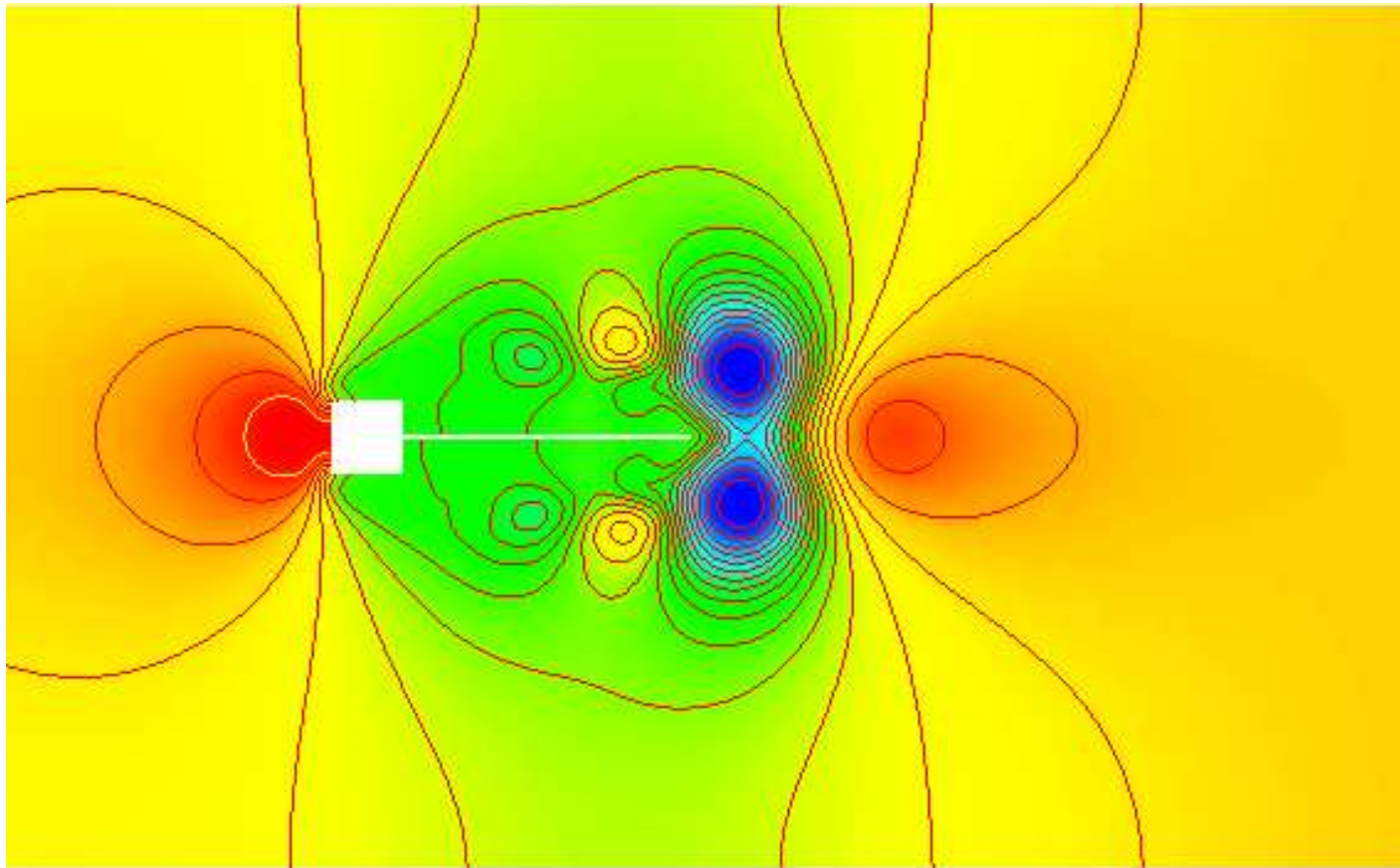


Abbildung 4: Pressure field $t = 0.5$



5. Results, III

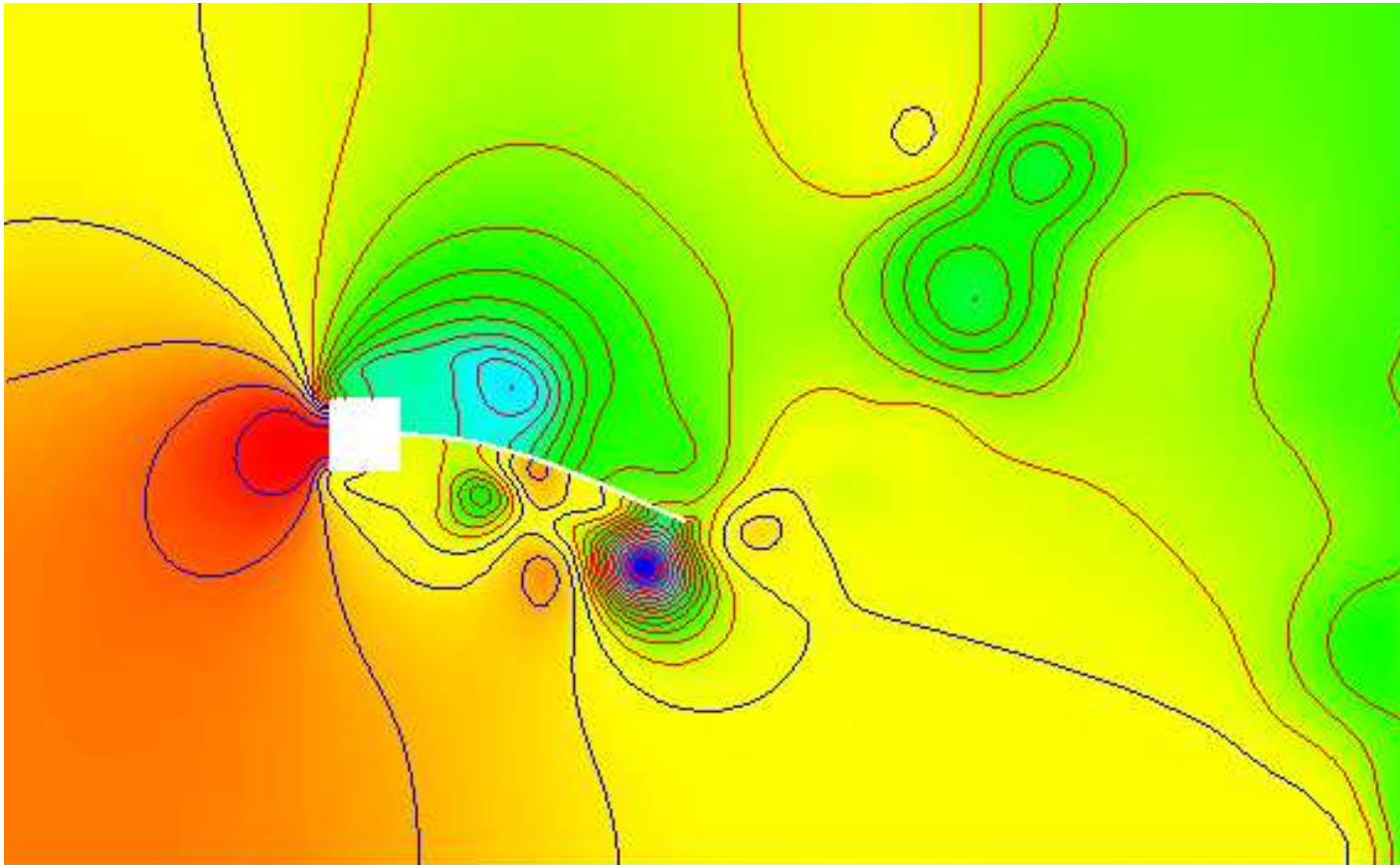


Abbildung 5: Pressure field $t = 10.12$



5. Results, IV

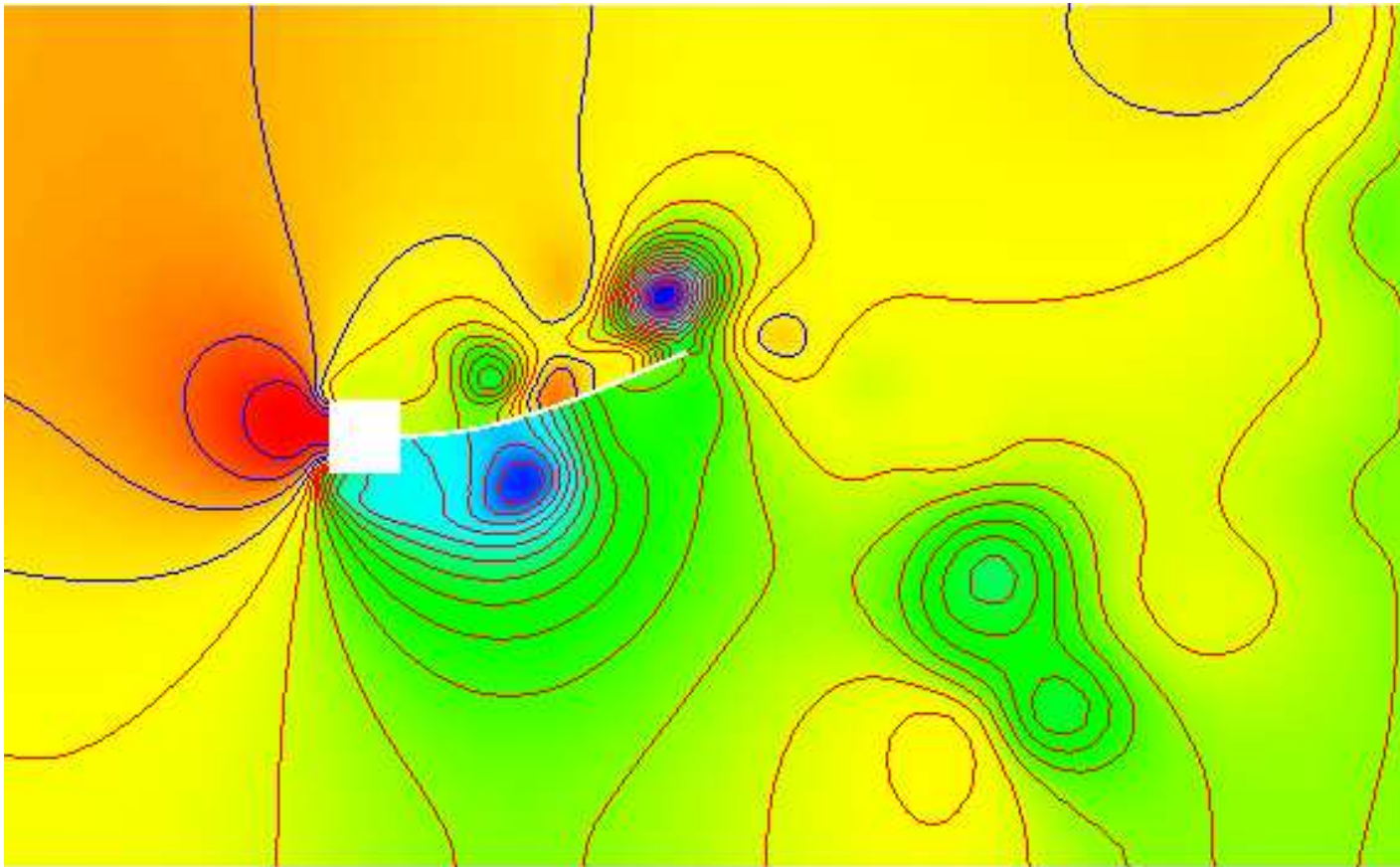


Abbildung 6: Pressure field $t = 10.28$