

***Multilevel
discontinuous Galerkin methods
on locally refined meshes***

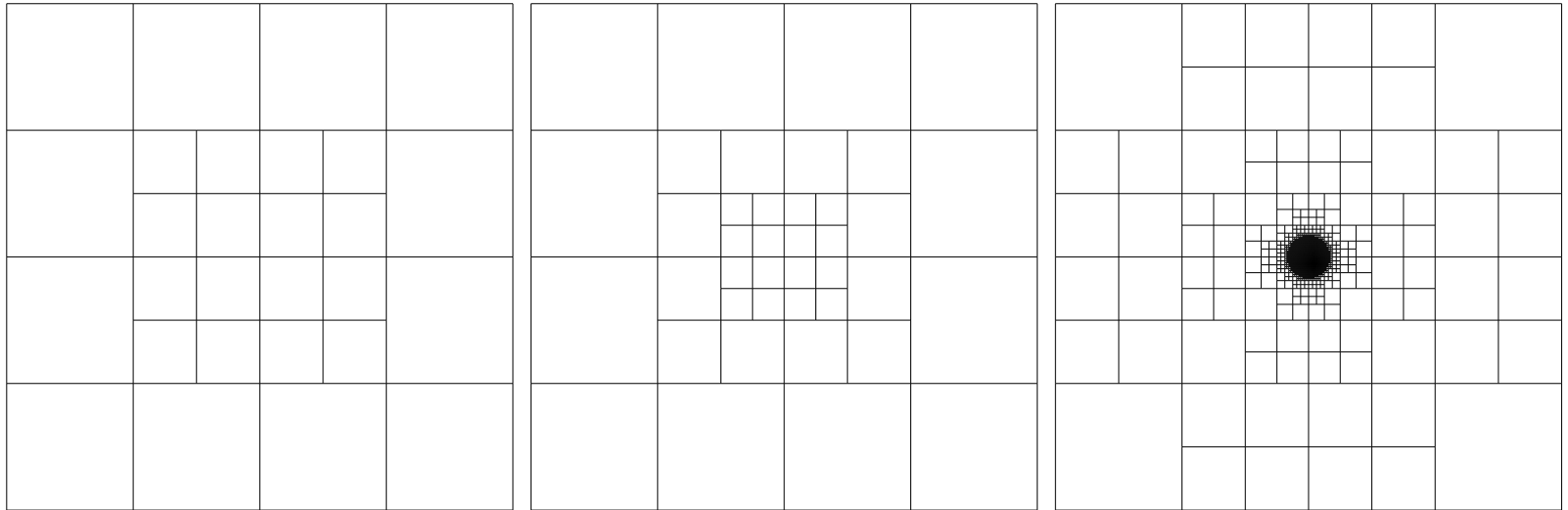
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Hierarchies

- ◆ Grids: $\mathbb{T}_0 \subset \mathbb{T}_1 \cdots \subset \mathbb{T}_L$



- ◆ Finite element spaces

$$V_0 \subset V_1 \subset \cdots \subset V_L$$

of piecewise polynomial functions on \mathbb{T}_ℓ .

Components of Multigrid

- ◆ Smoothers on each level
 - ◆ Jacobi
 - ◆ Gauß-Seidel
- ◆ Transfer between levels
 - ◆ Injection
 - ◆ L^2 -projection
- ◆ Solver on the coarsest level
 - ◆ small matrices

Complexity

- ◆ Multigrid is the only preconditioner for elliptic problems with optimal complexity
 - ◆ holds on uniformly refined meshes
 - ◆ Effort for level L :

$$n_L + \frac{n_{L-1}}{4} + \frac{n_{L-2}}{16} + \dots \leq \frac{4}{3}n_L + \mathcal{O}(1)$$

- ◆ Aim: maintain optimal complexity on locally refined meshes
 - ◆ Mesh sizes do not grow geometrically
 - ◆ Cure: local smoothing

What is local smoothing?

- ♦ Restrict the smoother to the refined part of a level
- ♦ Tasks:
 - ① Identify a “refined” subspace of the finite element space
 - ② Define a smoothing operator on this space
 - ③ Maintain consistency with the whole space

Continuous finite elements

- ◆ Bilinear forms have support on grid cells

$$a(u, v) = \sum_T a_T(u, v)$$

- ◆ Finite element functions overlap several cells
- 👉 Degrees of freedom on faces/vertices

Discontinuous Galerkin

- ♦ Finite element basis with support inside cells
- 👉 No topological coupling between cells

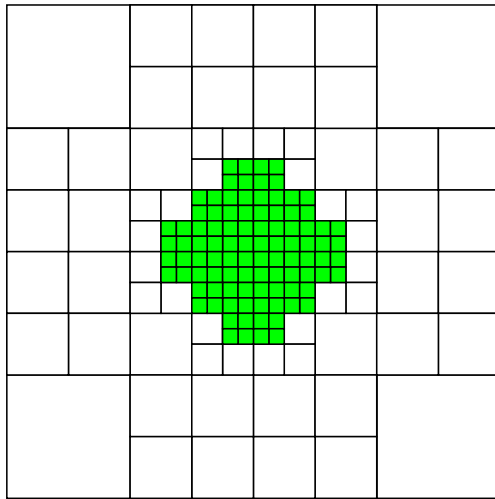
$$V = \bigoplus V_T$$

- ♦ Bilinear forms have support on cells and faces

$$a(u, v) = \sum_T a_T(u, v) + \sum_F a_F(u, v)$$

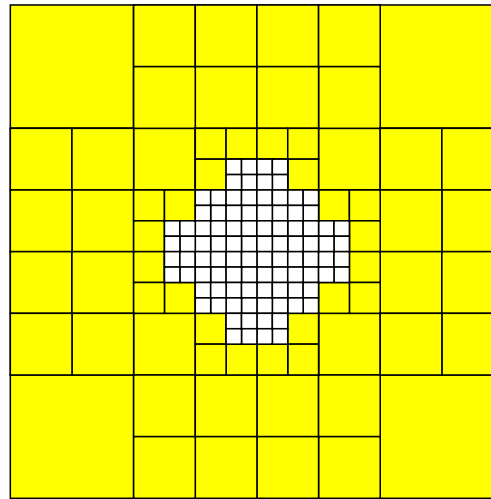
Splitting into subspaces

- ◆ Local smoothing: find $V_L = V_L^S \oplus V_L^L$



V_L^S

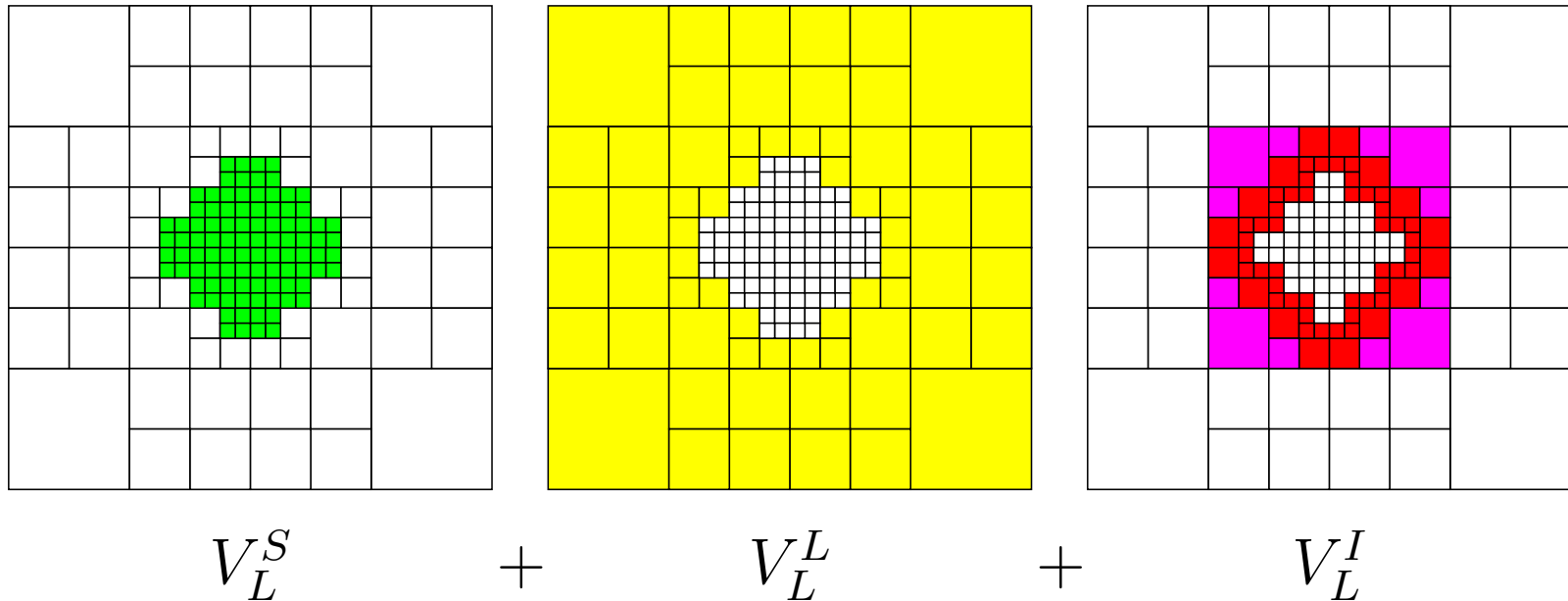
+



V_L^L

Splitting into subspaces

- ◆ Local smoothing: find $V_L = V_L^S \oplus V_L^L$



- 👉 Continuous elements require smoothing on $V_L^S + V_L^I$.

Splitting of the Matrix

$$V_\ell = V_\ell^S \oplus V_\ell^L,$$

where V_ℓ^S and V_ℓ^L have support in \mathbb{T}_ℓ^S and \mathbb{T}_ℓ^L , resp.

$$\begin{pmatrix} A_\ell^S & A_\ell^{SL} \\ A_\ell^{LS} & A_\ell^L \end{pmatrix} \begin{pmatrix} u_\ell^S \\ u_\ell^L \end{pmatrix} = \begin{pmatrix} b_\ell^S \\ b_\ell^L \end{pmatrix}$$

Matrices A_ℓ^{SL} and A_ℓ^{LS} correspond to refinement edge

Local smoothing and grid transfer

$$x_\ell^1 = x_\ell^0 - S_\ell^{-1}(b_\ell - A_\ell x_\ell^0)$$

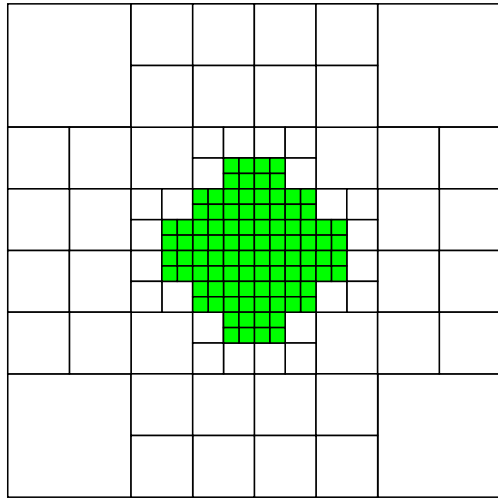
- ◆ Smoothing is restricted to the refined part V_ℓ^S

$$S_\ell = \begin{pmatrix} S_\ell^S & 0 \\ 0 & 0 \end{pmatrix}$$

- ◆ Prolongation is embedding
- ◆ Restriction acts as identity on V_ℓ^L

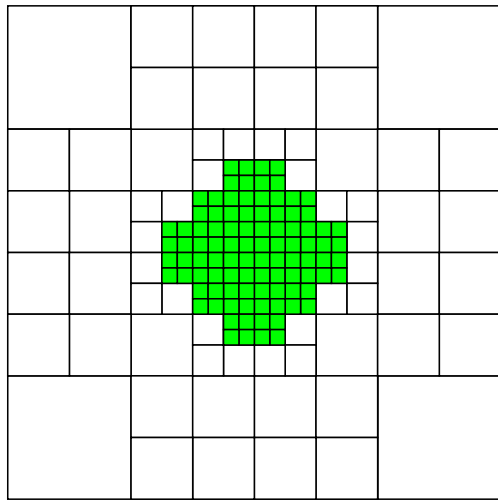
$$I_\ell^{\ell-1} u_\ell = I_\ell^{\ell-1;S} u_\ell^S + u_\ell^L$$

The local V-cycle

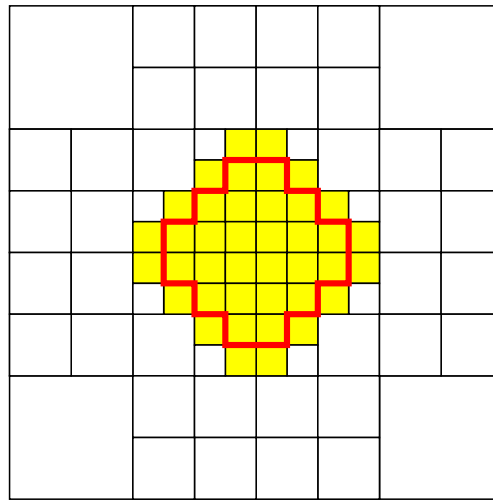
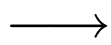


pre-smoothing

The local V-cycle

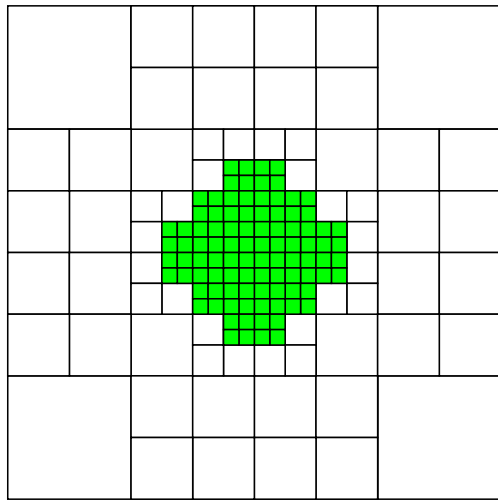


pre-smoothing

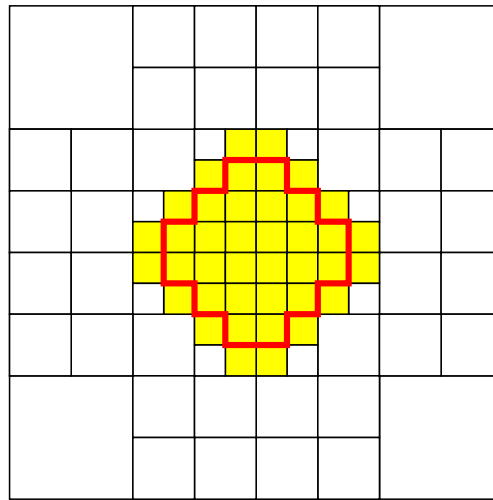
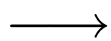


residual

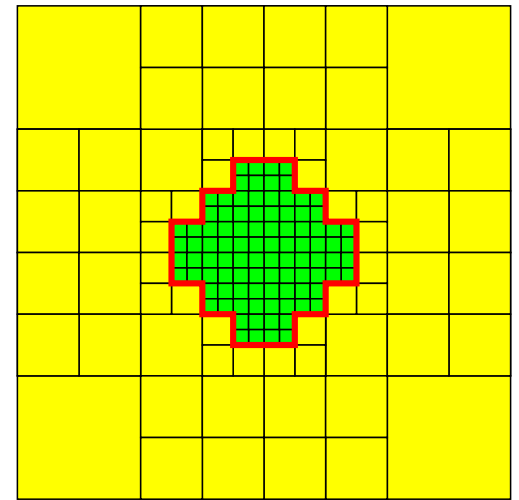
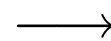
The local V-cycle



pre-smoothing



residual



post-smoothing

The local V-cycle

- ◆ Pre-smoothing:

$$R_\ell(u_\ell, b_\ell) = \begin{pmatrix} u_\ell^S + S_\ell^S (b_\ell^S - A_\ell^S u_\ell^S) \\ 0 \end{pmatrix}$$

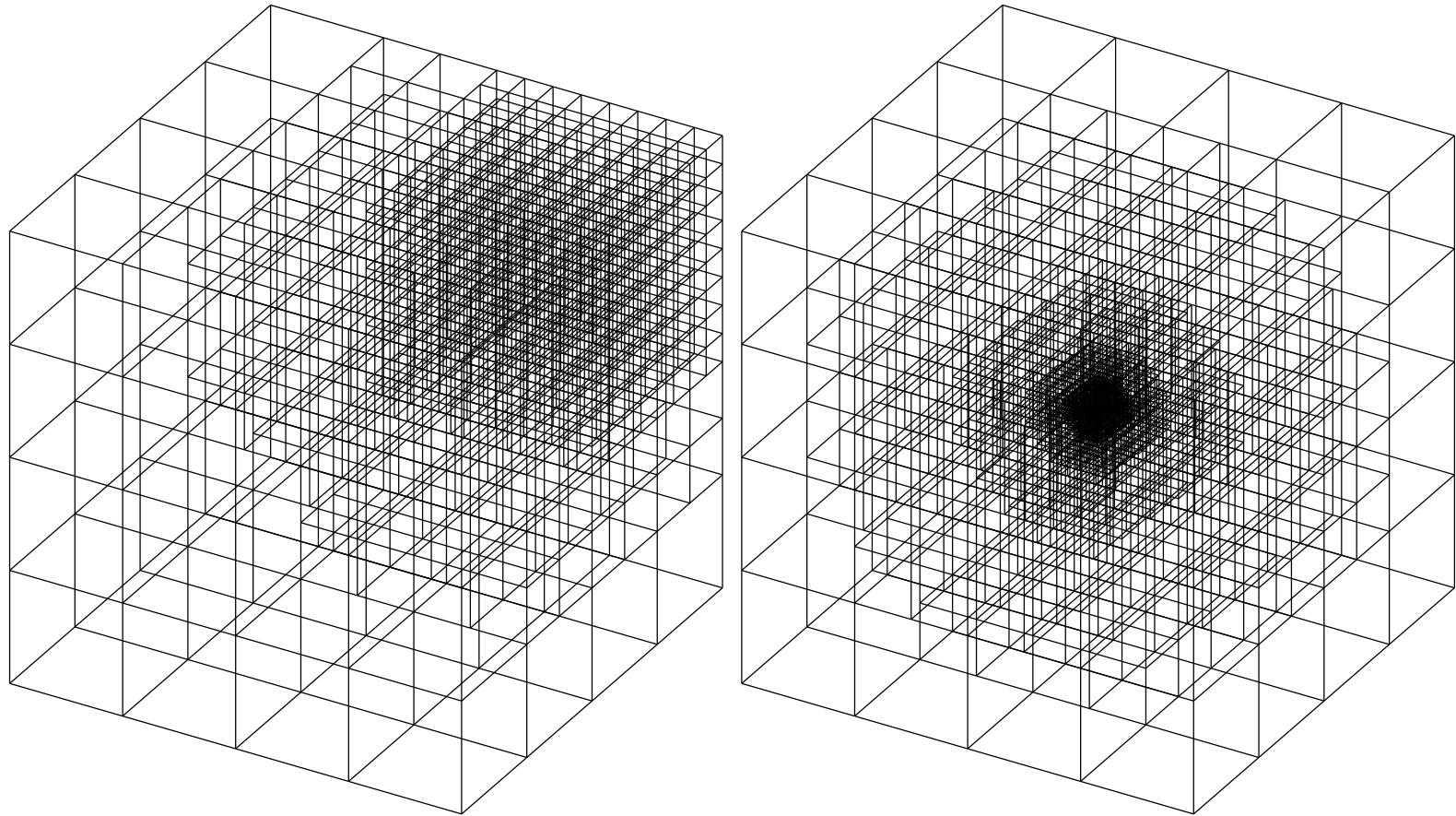
- ◆ Coarse grid correction

$$b_{\ell-1} = I_\ell^{\ell-1;S} (b_\ell^S - A_\ell^S u_\ell^{1;S}) + b_\ell^L - A_\ell^{LS} u_\ell^{1;S}$$

- ◆ Post-smoothing

$$\tilde{b}_\ell^S = b_\ell^S - A_\ell^{SL} u_c^L$$

3D meshes



Results in 3D, cg

L	\mathbb{Q}_1 -elements						\mathbb{Q}_2 -elements					
	global		octant		ball		global		octant		ball	
	n_{10}	r	n_{10}	r	n_{10}	r	n_{10}	r	n_{10}	r	n_{10}	r
1	7	.04					11	.11				
2	11	.11	10	.10	11	.11	14	.19	13	.17	14	.19
3	12	.13	13	.16	13	.16	15	.21	15	.21	15	.20
4	12	.14	14	.17	12	.14	15	.21	16	.23	14	.18
5	13	.15	13	.17	12	.13	15	.21	15	.21	13	.16
6	13	.15	13	.16	12	.15			15	.20	14	.17
7			13	.15	12	.14					14	.18
8					12	.14					14	.17
9					12	.13					13	.16

Conclusions for discontinuous Galerkin

- ◆ Ingredients
 - ◆ Level matrices built of cells strictly on level ℓ
 - ◆ Smoothers for these matrices
 - ◆ Interface matrices on refinement edge
- ◆ Multigrid works as usually
- 👉 No loss of performance
- 👉 Optimal complexity even for very locally refined meshes

Obstacles for continuous Galerkin

- ◆ Degrees of freedom *on* the interface
- ☞ Overlapping spaces (where to put V_ℓ^I ?)
- ◆ Bilinear elements work moderately well
- ✗ Higher order did *not* work (implementation?)

Solutions for continuous Galerkin

- ◆ local smoothing
 - ◆ Smoothing on $V_\ell^S \cup V_\ell^I$
 - 👉 Level matrices include interface part
- ◆ global coarsening
 - ◆ Every level covers the whole domain
 - 👉 A_L is the global matrix
 - ◆ For \mathbb{T}_{L-1} , remove finest level and coarsen as many cells once as mesh smoothing allows
 - ◆ Do not coarsen beyond level zero

Effects on structures

- ✘** Hanging nodes in level spaces
- ✘** Smoothers must handle hanging nodes
- ✘** Every grid cell may belong to several levels
 - ☞** Levels can only be determined globally
 - ☞** Several numberings of MGDoFs on each cell
- ☞** New structures and considerable effort needed

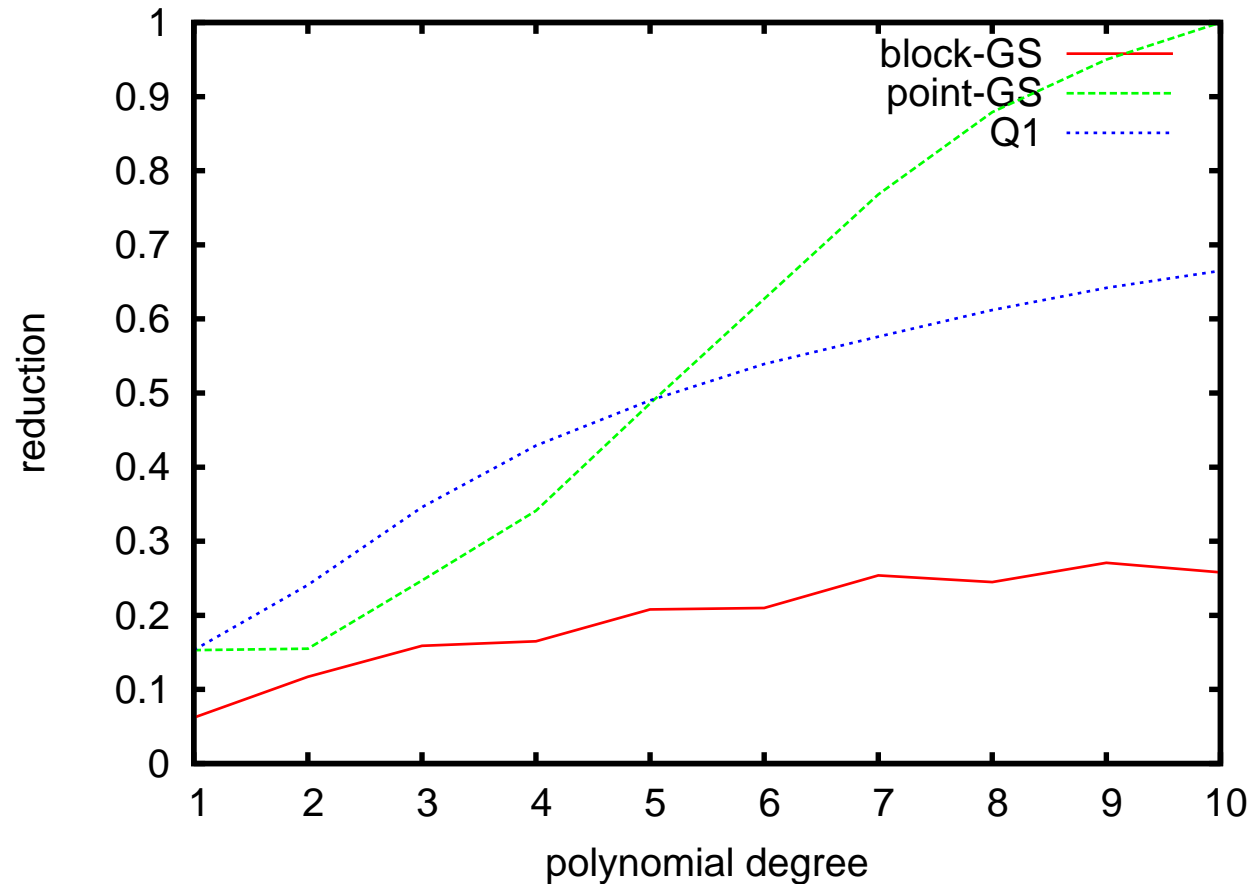
Remarks on Higher Order Elements

- ◆ Q_k elements of arbitrary order
- ◆ based on equidistant Lagrange interpolation
- 👉 condition number of cell matrices large (Q_{10} : 10^7)

Block smoothers for higher order elements

- ◆ Cell matrices of DG methods correspond to well-posed BVP
- 👉 cell matrices are invertible
- 👉 block Gauß-Seidel smoother
- 👉 effect of ill conditioned cell matrix removed

Convergence for higher order elements



- ◆ behavior not yet analyzed (K./Karakashian in prep.)
- ◆ Dependence on σ suggests faster growth